

HILL'S SURFACES AND THEIR THETA FUNCTIONS¹

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Preface. In this paper we continue the investigations begun in McKean-Trubowitz [1976] of infinite-genus hyperelliptic Riemann surfaces S which are constructed from the spectrum of a Hill's operator. Let q be a real infinitely differentiable function of $0 \leq \xi < 1$ of period 1. The Hill's operator is $Q = -d^2/d\xi^2 + q(\xi)$. The periodic and antiperiodic eigenfunctions of Q determine an infinite spectrum $\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots \uparrow \infty$ of simple or double eigenvalues. S is formed by cutting two copies of the number sphere along the so-called intervals of instability marked off by such pairs of simple eigenvalues $\lambda_{2n-1} < \lambda_{2n}$ as may occur. S is a hyperelliptic surface of genus g ($< \infty$) equal to the number of such pairs. The purpose of this paper is to develop some of the function theory of S in the case $g = \infty$ with special attention to differentials of the first kind, the Jacobian variety, and the Riemann theta function. McKean-Trubowitz [1976] introduced a Hilbert space of differentials of the first kind closely connected with the interpolation of certain classes of entire functions, defined the Jacobi map for divisors in "real position", and constructed the "real part" of the Jacobian variety. The present paper studies more refined Hilbert spaces of differentials; one such space of particular importance is populated by differentials with precisely " $2 \times (g = \infty) - 2$ " roots, just as in the classical case. The associated (infinite-dimensional) Jacobian variety and its theta function are also introduced. The basic properties of the latter include a variant of the Riemann vanishing theorem. A theta function formula of Baker [1897] and Its-Matveev [1975] is adapted to the present case and used to express the solution of the celebrated Korteweg-deVries equation with periodic initial data. Along the way, we prove period relations, derive an infinite-dimensional analogue of Jacobi's identity for the theta function, embed S in its Jacobian variety, and prove the easy half of Abel's theorem.

To the best of our knowledge the only previous work on transcendental hyperelliptic function theory is that of Hornich [1933], [1935], [1939] and Myrberg [1943], [1945]. These papers contain some discussion of square-summable differentials of the first kind and their period relations. The

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infinite-dimensional Jacobian variety and its theta function are introduced for the first time in the present paper.

1. Preliminaries. *Hill's operator:* Q denotes the Hill's operator $-d^2/d\xi^2 + q(\xi)$ with potential q in C_1^∞ , the space of smooth real-valued functions of period 1. $\int_0^1 q(\xi) d\xi$ is assumed to vanish with a view to the simplicity of certain estimates. Let $y_1(\xi, \lambda)$ and $y_2(\xi, \lambda)$ be the solutions of $Qy = \lambda y$ with $y_1(0, \lambda) = y_2(0, \lambda) = 1$ and $y_1'(0, \lambda) = y_2'(0, \lambda) = 0$. The discriminant $\Delta(\lambda)$ is defined to be $y_1(1, \lambda) + y_2'(1, \lambda)$. Most of the following basic material can be found in Levitan-Sargsjan [1975], Magnus-Winkler [1966], and McKean-Trubowitz [1976, §1].

The periodic spectrum is the sequence

$$\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \cdots \uparrow \infty$$

of simple or double eigenvalues of Q arising from eigenfunctions of period 2 (or 1); equality means that $\lambda_{2n-1} = \lambda_{2n}$ has a 2-dimensional eigenspace. For simplicity, let the periodic spectrum be purely simple, i.e., $\lambda_{2n-1} < \lambda_{2n}$ ($n \geq 1$).

The estimate

$$\lambda_{2n-1}, \lambda_{2n} = n^2\pi^2 + O(1/n^2)$$

is noted for future use. The lowest eigenvalue λ_0 is simple and f_0 , the corresponding eigenfunction, is root-free and of period 1. The eigenfunctions f_{2n-1}, f_{2n} corresponding to $\lambda_{2n-1}, \lambda_{2n}$ have n roots apiece in $[0, 1)$ and are periodic when n is even and antiperiodic when n is odd. The f_n 's are normalized by $\int_0^1 f_n^2 d\xi = 1$. The roots of $\Delta^2(\lambda) - 4 = 0$ coincide with the periodic spectrum: in fact $\Delta(\lambda_0) = 2$ and $\Delta(\lambda_{2n-1}) = \Delta(\lambda_{2n}) = 2(-1)^n$ ($n \geq 1$). The intervals $(-\infty, \lambda_0), (\lambda_1, \lambda_2), (\lambda_3, \lambda_4), \dots$ are intervals of instability, so-called because no solution of $Qy = \lambda y$ is bounded on the line if λ lies in such an interval. The widths $l_n = \lambda_{2n} - \lambda_{2n-1}$ ($n \geq 1$) are rapidly decreasing by a theorem of Hochstadt [1963].

The roots μ_n ($n \geq 1$) of $y_2(1, \mu) = 0$ coincide with the eigenvalues of Q arising from eigenfunctions that vanish at $\xi = 0$ and $\xi = 1$. They interlace the periodic spectrum as follows:

$$\lambda_0 < \lambda_1 < \mu_1 < \lambda_2 < \lambda_3 < \mu_2 < \lambda_4 < \cdots$$

and are designated as the *tied spectrum*. The normalized eigenfunction corresponding to μ_n is $y_2(\xi, \mu_n) \times [\int_0^1 y_2^2(\xi, \mu_n) d\xi]^{-1/2}$; the number

$$\int_0^1 y_2^2(\xi, \mu_n) d\xi = y_2'(1, \mu_n) y_2(1, \mu_n)$$

is the n th norming constant. The following estimates of y_1 and y_2 will often be used:

$$y_1(\xi, \lambda) = \cos \sqrt{\lambda} \xi + \frac{\sin \sqrt{\lambda} \xi}{2\sqrt{\lambda}} \int_0^\xi q d\eta + O(\lambda^{-1})$$

$$y_2(\xi, \lambda) = \frac{\sin \sqrt{\lambda} \xi}{\sqrt{\lambda}} - \frac{\cos \sqrt{\lambda} \xi}{2\sqrt{\lambda}} \int_0^\xi q d\eta + O(\lambda^{-3/2})$$

² · means $\partial/\partial\mu$.

for $\lambda \uparrow \infty$, and

$$y_1(\xi, \lambda) = \cos \sqrt{\lambda} \xi [1 + o(1)]$$

$$y_2(\xi, \lambda) = \frac{\sin \sqrt{\lambda} \xi}{\sqrt{\lambda}} [1 + o(1)]$$

for $\lambda \downarrow -\infty$. These estimates can be differentiated with respect to x and/or λ and are uniform on $0 \leq x \leq 1$. The functions $y_2(1, \lambda)$ and $\Delta(\lambda)$ are entire of order $1/2$ and type 1. They can be expressed as

$$y_2(1, \mu) = \prod_{n \geq 1} \frac{\mu_n - \mu}{n^2 \pi^2}$$

and

$$\Delta(\lambda) = 2(\lambda_0 - \lambda) \prod_{n \text{ even}} \frac{(\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}{n^4 \pi^4} + 2,$$

$$\Delta(\lambda) = 2 \prod_{n \text{ odd}} \frac{(\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}{n^4 \pi^4} - 2,$$

respectively; in particular, μ_n ($n \geq 1$) determines $y_2(1, \lambda)$, while λ_0 , λ_{2n-1} , λ_{2n} (n even) or λ_{2n-1} , λ_{2n} (n odd) determine $\Delta(\lambda)$.

Spaces of entire functions. Two real Hilbert spaces of entire functions are introduced for future use. First, $I^{1/2}$ is the Hilbert space of entire functions ϕ , real on the line, of order $\leq 1/2$ and type ≤ 1 , for which

$$I^{1/2}[\phi] = \int_0^\infty |\phi(\lambda)|^2 \lambda^{1/2} d\lambda < \infty.$$

$I^{1/2}(\phi, \psi) = \int_0^\infty \phi \psi \lambda^{1/2} d\lambda$ is the inner product of the space $I^{1/2}$. Let $\mu_n \in [\lambda_{2n-1}, \lambda_{2n}]$ ($n \geq 1$) be any tied spectrum and let $y_2(1, \lambda) = \prod (n^2 \pi^2)^{-1} (\mu_n - \lambda)$. The functions $y_2(1, \lambda) [(\lambda - \mu_n) y_2(1, \mu_n)]^{-1}$ ($n \geq 1$) are in $I^{1/2}$. Let $\phi \in I^{1/2}$. Then $\sum_{n \geq 1} |\phi(\mu_n)|^2 n^2$ is comparable³ to $I^{1/2}[\phi]$, and⁴

$$\phi(\lambda) = \sum_{n \geq 1} \phi(\mu_n) \frac{y_2(1, \lambda)}{(\lambda - \mu_n) y_2(1, \mu_n)}$$

$$= \sum_{n \geq 1} \phi(\mu_n) \prod_{i \neq n} \frac{\mu_i - \lambda}{\mu_i - \mu_n},$$

i.e., ϕ can be interpolated off μ_n ($n \geq 1$). The same is true of the Hilbert space $I^{3/2} \subset I^{1/2}$ defined by

$$I^{3/2}[\phi] = \int_0^\infty |\phi|^2 \lambda^{3/2} d\lambda < \infty$$

with one modification: now it is the sum $\sum_{n \geq 1} |\phi(\mu_n)|^2 n^4$ which is comparable to $I^{3/2}[\phi]$; see McKean-Trubowitz [1976, §5] for more information.

The Hill's surface. The Hill's surface S for the potential q is constructed by

³Here, *comparable* means $aI^{1/2}[\phi] < \sum_{n \geq 1} |\phi(\mu_n)|^2 n^2 < bI^{1/2}[\phi]$, with $a, b > 0$, independently of $\phi \in I^{1/2}$; the word is used in the same way below.

⁴The sums converge in norm and uniformly on compact subsets of \mathbb{C} .

cutting two copies of the Riemann sphere along the intervals of instability and attaching all the lower lips on one sphere to the corresponding upper lips on the other and *vice versa*. S is the Riemann surface of the transcendental hyperelliptic irrationality $R(\lambda) = \sqrt{(1/4)\Delta^2(\lambda) - 1}$. In particular, S has infinite genus. Figure 1 represents a topological model of S on which a canonical homology basis $A_1, B_1, A_2, B_2, \dots$ has been drawn.

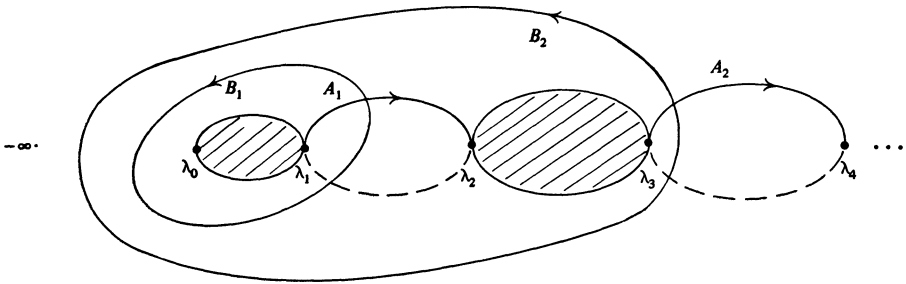


FIGURE 1

Potentials with a common periodic spectrum. Fix $q_0 \in C_1^\infty$ and let M be the space of all potentials $q \in C_1^\infty$ with the same periodic spectrum λ_n ($n \geq 0$) as q_0 . M is compact in the topology it inherits from C_1^∞ . Slit the n th interval of instability $(\lambda_{2n-1}, \lambda_{2n})$ and for $q \in M$, place $\mu_n \in [\lambda_{2n-1}, \lambda_{2n}]$ on the upper or the lower lip according as the radical $R(\mu_n)$ is positive or negative. A potential q in M is mapped thereby to the point $p = (p_1, p_2, \dots)$ of the torus of Figure 2 determined by $p_n = (\mu_n, R(\mu_n))$ ($n \geq 1$). This map is a diffeomorphism onto. Thus, M is a torus naturally diffeomorphic to a product of circles sitting inside the infinite product of S with itself; see McKean-Trubowitz [1976, §4] for more information.

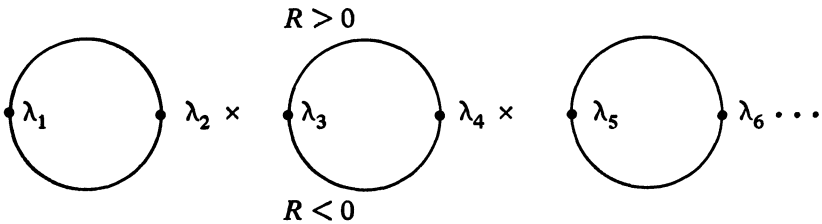


FIGURE 2

For fixed real λ , let the vector field \mathbf{X} be defined by $\mathbf{X}q(\xi) = (d/d\xi)(\partial\Delta(\lambda)/\partial q(\xi))$. Then the flow $e^{t\mathbf{X}}$ generated by solving $\partial q/\partial t = \mathbf{X}q$ preserves M , and any two of these flows commute. Observe that the map $q(\xi) \rightarrow \mathbf{X}q(\xi)$ is not local. The \mathbf{X} 's are Hamiltonian vector fields on M with respect to the Poisson bracket

$$\{F, G\} = \int_0^1 (\partial F/\partial q(\xi))(d/d\xi)(\partial G/\partial q(\xi)) d\xi$$

between functions F and G of q , i.e., \mathbf{X} is the gradient $\partial/\partial q$ of the Hamiltonian $\Delta(\lambda)$ followed by the skew symmetric operator $d/d\xi$. The tangent space to M at a point q is the span of the functions $\mathbf{X}q$; see §§3, 8 and

9 of McKean-Trubowitz [1976] for a complete discussion of these matters.

There is an important hierarchy of local vector fields V_n ($n \geq 1$) on M which is constructed from the semigroup e^{-tQ} . The trace

$$\text{tr}(e^{-tQ}) = \sum_{n>0} e^{-\lambda_n t}$$

has the asymptotic expansion⁵

$$\text{tr}(e^{-tQ}) \sim \frac{1}{\sqrt{\pi t}} \sum_{n>0} \frac{(-t)^n H_{n-1}}{(2n-3) \cdot \dots \cdot 3 \cdot 1} \quad (t \downarrow 0)$$

in which $H_{-1} = 1$ and H_n ($n \geq 0$) is an integral over the period $0 \leq \xi < 1$ of a universal polynomial in q and its derivatives. The vector field V_n ($n \geq 1$) is defined by

$$V_n q(\xi) = \frac{d}{d\xi} \frac{\partial H_n}{\partial q(\xi)}$$

and induces a local Hamiltonian flow e^{tV_n} on M . The first three are $V_1 q = q'$, $V_2 q = 3qq' - \frac{1}{2}q'''$, and $V_3 q = (15/2)q^2q' - 5q'q'' - (5/2)qq''' + (1/4)q''''$, the flow induced by V_2 being equivalent to the well-known Korteweg-deVries equation $\partial q / \partial t = 3qq' - (1/2)q'''$. The general rule is $V_n q = (qD + Dq - (\frac{1}{2})D^3)(\partial H_{n-1} / \partial q)$. See Gardner, Greene, Kruskal-Miura [1974] and McKean-Trubowitz [1976: 147-148] for more information; note that

$$V_i q = \sum_{n>0} p_i(\lambda_{2n}) \frac{d}{d\xi} \frac{\partial \Delta(\lambda_{2n})}{\partial q(\xi)} \quad (i \geq 0)$$

where p_i is a polynomial of degree i depending upon M but not upon q ; see McKean-Trubowitz [1976: 176] for a proof.

Differentials, Abelian sums, and the real part of the Jacobian variety for S. A differential $d\Phi$ of the first kind on S is of the form $\phi d\lambda / R$, ϕ being a suitable entire function. Let $I = I^{3/2}$ and introduce the subspaces (a) J , and (b) K , defined by the requirements (a) $\phi(\mu_n)$ is rapidly decreasing, and (b) $K[\phi] = \sum_{n>1} |\phi(\mu_n)|^2 I_n^{-2} < \infty$; μ_n ($n \geq 1$) is any tied spectrum; it does not matter which one is employed since $\phi^*(\lambda) = O(\lambda^{-3/2})$ for $\phi \in I$. Notice that such a differential is completely determined by its real periods

$$A_n(\phi) = \oint_{A_n} (\phi/R) d\lambda = 2 \int_{\lambda_{2n-1}}^{\lambda_{2n}} (\phi/R) d\lambda \quad (n \geq 1)$$

since $A_n(\phi) = 2\phi(\mu_n) \int_{\lambda_{2n-1}}^{\lambda_{2n}} R^{-1} d\lambda$ for suitable $\mu_n \in [\lambda_{2n-1}, \lambda_{2n}]$, so that $A_n(\phi) = 0$ ($n \geq 1$) implies $\phi = 0$ by interpolation. For future use, note that the period $B_n(\phi)$ of the differential $d\Phi$ is the integral of $d\Phi$ around the oriented cycle B_n , i.e., counterclockwise about the segment $[\lambda_0, \lambda_{2n-1}]$.

The next topic is the dual space I^\dagger of I . To begin with, the periods A_n belong to I^\dagger , and by the previous remark they span I^\dagger ; in fact, any $x \in I^\dagger$ is

⁵ $(2n-3) \cdot \dots \cdot 3 \cdot 1$ is construed as 1 if $n = 0$ or 1.

uniquely expressible as $\sum_{n \geq 1} x_n A_n$ with $\sum_{n \geq 1} x_n^2 n^{-2} < \infty$.⁶ Now let $o_n = (\lambda_{2n-1}, 0)$ ($n \geq 1$) and let $p_n = (\mu_n, R(\mu_n))$ ($n \geq 1$) be any sequence of points in "real position" on S , meaning that $\mu_n \in [\lambda_{2n-1}, \lambda_{2n}]$. Then the Abelian sum $x(\phi) = \sum_{n \geq 1} \int_{o_n}^{p_n} \phi/R d\lambda$ defines an element of I^\dagger provided the paths of integration satisfy $\sum_{n \geq 1} m_n^2 n^{-2} < \infty$, m_n being the number of full revolutions that the path from o_n to p_n makes about the circle on which it lies. The sums obtained in this way fill out all of I^\dagger . Now let $L_I \subset I^\dagger$ be the lattice of periods $\sum_{n \geq 1} m_n A_n$ with m_n as above. Then $I^\dagger/L_I = \mathfrak{S}$ is compact and in a very natural sense diffeomorphic to M , especially, the map $q \rightarrow (p_1, p_2, \dots) \rightarrow y = (x \text{ modulo } L_I) \in \mathfrak{S}$ is 1:1. \mathfrak{S} is the "real part" of the Jacobian variety for S ; the nomenclature arises from the fact that the A -periods (B -periods) of a hyperelliptic surface with real branch points are real (purely imaginary), so that, for such a surface of genus $g < \infty$, the full Jacobian variety $C^g/(\text{periods})$ naturally breaks up into a "real part" $R^g/(A\text{-periods})$ and an "imaginary part" $R^g/\sqrt{-1}$ (B -periods). The same procedure can be applied to J and K : $\sum_{n \geq 1} x_n A_n \in J^\dagger$ if and only if x_n is of polynomial growth as $n \uparrow \infty$ and with this restriction on the winding numbers m_n , $J^\dagger/L_J = \mathfrak{S}$ with a selfevident notation; similarly, $\sum_{n \geq 1} x_n A_n \in K^\dagger$ if and only if $\sum_{n \geq 1} x_n^2 (nl_n)^2 < \infty$, and with this restriction on the winding numbers, $K^\dagger/L_K = \mathfrak{S}$, too. Summarizing,

$$I^\dagger/L_I = J^\dagger/L_J = K^\dagger/L_K = \mathfrak{S} \cong M.$$

The final point to be discussed is the inversion of the Jacobi map $q \rightarrow p \rightarrow x \in \mathfrak{S}$. Let $y_2(1, \lambda)$ and μ_n be computed for q translated in the amount $0 < \xi < 1$. Then⁷ $\partial\Delta(\lambda)/\partial q(\xi) = y_2(1, \lambda) = \prod_{n \geq 1} (n\pi)^{-2} (\mu_n - \lambda)$, and⁸

$$\frac{d}{d\xi} \frac{\partial\Delta(\lambda)}{\partial q(\xi)} = - \sum_{n \geq 1} 2R(\mu_n) \frac{y_2(1, \lambda)}{(\lambda - \mu_n)y_2(1, \mu_n)},$$

the estimate⁹ $R(\mu_n) = O(n^{-1})l_n$ being employed to justify the differentiation. Now let $\psi = (d/d\xi) \partial\Delta(\lambda)/\partial q(\xi)$ for fixed $0 < \xi < 1$. Then $\psi \in K$: in fact, $\psi \in I^{3/2}$ by interpolation, and $K[\psi] < \infty$ since $\psi(\mu_n) = -2R(\mu_n) = O(n^{-1})l_n$. Now, the inverse map is easy to write down. Let $x \in K^\dagger$ and let the potential with tied spectrum $\mu_n = \lambda_{2n-1}$ ($n \geq 1$) be distinguished as the origin of M . Then $Xq = x(\psi)$, qua function of $0 < \xi < 1$, induces a vector field X on M and the inversion of the map

$$q \rightarrow p \rightarrow x \in \mathfrak{S} = K^\dagger/L_K$$

is achieved by the rule¹⁰ $e^{X(\text{origin})} = q$. This may be regarded as an exponential map of K^\dagger , the tangent space of \mathfrak{S} , onto $M \cong \mathfrak{S}$; it is emphasized that unlike a classical exponential map the present map is *never* locally 1:1 as

⁶ $|\int_{\lambda_{2n-1}}^{\lambda_{2n}} R^{-1} d\lambda|$ is comparable to n for $n \uparrow \infty$; see McKean-Trubowitz [1976: 199]. By this remark $K[\phi]$ is comparable to $\sum_{n \geq 1} |A_n(\phi)|^2 |nl_n|^{-2}$.

⁷ McKean-Trubowitz [1976: 159].

⁸ $d\mu_n/dx = 2R(\mu_n)/y_2(1, \mu_n)$; see McKean-Trubowitz [1976: 167].

⁹ The mean-value theorem is used, together with the fact that $\Delta^{-1}(\lambda) = (2\lambda)^{-1} \cos \sqrt{\lambda} + O(\lambda^{-3/2}) (\lambda \uparrow \infty)$.

¹⁰ For a general vector field V the flow e^{tV} on M is defined by solving the equation $\partial q/\partial t = Vq$ for time t .

the unit ball of K^\dagger contains infinitely many periods from L_K . This is just as it should be: $M \cong \mathfrak{S}$ is compact while the unit ball of K^\dagger is not. See McKean-Trubowitz [1976: 213–218] for more about I^\dagger/L_I and J^\dagger/L_J ; the quotient K^\dagger/L_K is not treated fully there, but the necessary details are easily supplied.

2. General introduction. In this paper we study the transcendental hyperelliptic Hill’s surface S , introduce the associated theta function and establish some of its basic properties. An outline of our main results is given below.

Differentials. Let H be the Hilbert space of all differentials $d\Phi = (\phi/R) d\lambda$ with

$$H[\phi] = \sqrt{-1} \int_S d\Phi \wedge \overline{d\Phi} = 4 \int_C \left| \frac{\phi}{R} \right|^2 d \text{area} < \infty,$$

ϕ entire and real on the line. In the finite genus case, apart from the reality condition, H is exactly the space of differentials of the first kind. In the present case, H is good for some purposes while J and K are better suited to others. It turns out that the most natural space of differentials is $K + \sqrt{-1} H$; this will be clarified below. We now summarize the facts concerning H which appear in §§3, 5 and 6.

To begin with $H \subset I^{1/2}$ and

$$\pi H[\phi] = 4 \int_0^\infty b^{-2} \int_{-\infty}^\infty \left| \int_a^{a+b} d\Phi \right|^2 da db,$$

from which it follows that

$$L[\phi] \equiv \sum_{n>1} |A_n(\phi)|^2 \log(1/l_n) \leq \text{a constant multiple of } H[\phi];$$

$l_n < 1$ is assumed for simplicity. Conversely, if $\phi \in I^{3/2}$ and $L[\phi] < \infty$, ϕ is in H ; in particular, $K \subset H$. The periods A_n, B_n ($n \geq 1$) are in H^\dagger , and you have the Riemann period relation

$$H[\phi] = -2\sqrt{-1} \sum_{n>1} A_n(\phi)B_n(\phi)$$

for differentials $d\Phi$ of class $H \cap I^{3/2}$. The same relation is true for differentials in $H \setminus I^{3/2}$ with a technical interpretation of the sum. Now, just as in the compact case, if $\phi \in H$ and either $A_n(\phi) = 0$ ($n \geq 1$) or $B_n(\phi) = 0$ ($n \geq 1$), then $\phi = 0$. Finally, H has a natural basis $1_j \in K$, ($j \geq 1$) such that $A_i(1_j) = 1$ or 0 according as $i = j$ or not, and

$$(\phi, 1_j)_H = -2\sqrt{-1} B_j(\phi) \quad (j \geq 1).$$

The next topic, the subject of §4, is the roots of a differential of class J . In the case of finite genus g , a differential of the first kind has exactly $2 \times g - 2$ roots. The same is true for differentials $d\Phi$ of class J on S once you determine the order to which $d\Phi$ vanishes at ∞ and can count infinite sets of points on S . At ∞ , the expansion

$$\phi(\lambda) \sim \lambda^{-3/2} \sqrt{-1} R(\lambda) \sum_{n>1} c_n(\phi) \lambda^{-n},$$

$$c_n(\phi) = \sum_{i>1} \frac{\phi(\lambda_{2i-1})}{y_{2i-1}^2(1, \lambda_{2i-1})} \lambda_{2i-1}^n \quad (n \geq 0)$$

holds on "nice" circles centered at λ_0 of radius $r \uparrow \infty$ with $|r - \lambda_i| \geq 1$ ($i \geq 1$). The important thing here is the estimate

$$|R(re^{\sqrt{-1}\theta})| \sim e^{\sqrt{r} |\sin(\theta/2)|}$$

on "nice" circles. Now, by analogy with the compact case, the parameter $\zeta = \lambda^{-1/2}$ can be used as a "local coordinate" at ∞ provided you approach ∞ via "nice" circles. Thus,

$$d\Phi \sim -2\sqrt{-1} \sum_{n>1} c_n \zeta^{2n} d\zeta$$

on "nice" circles, so it is natural to say that $d\Phi$ has a root of order $2m$ at ∞ if $c_j = 0$ ($j < m$) but $c_m \neq 0$; m may be infinite.

The finite roots of $d\Phi$ come in pairs of complementary points on S . Therefore, you need only count the roots of $d\Phi$ on one sheet, or what is the same thing, the roots of ϕ in the plane. To do this, introduce an intrinsic descending sequence of subspaces $I^{\infty-n}$ of $I^{1/2}$ defined by $I^{\infty-n}[\phi] = \int_0^\infty |\phi|^2 \lambda^{2n+1/2} d\lambda < \infty$ ($n \geq 0$). A function in $I^\infty \setminus I^{\infty-1}$ requires for its interpolation a point on every handle of S , e.g., λ_{2i-1} ($i \geq 1$), while a function in $I^{\infty-n} \setminus I^{\infty-(n+1)}$ requires only $\infty - n$ such points, i.e., any n , but no more, of the points λ_{2i-1} can be left out. We say that $d\Phi$ has $2\infty - 2m - 2$ finite roots on S if the $\infty - m - 1$ projections suffice to interpolate $I^{\infty-m-1}$ but not $I^{\infty-m}$. Of course the projected roots can be multiple and need not be real so that the Lagrange-type interpolating sums introduced in §1 must be replaced by a more general prescription. There is only one natural way to do this, and it is spelled out in §4. With these preliminaries, a differential $d\Phi$ of class J with a root of multiplicity $2m < \infty$ at ∞ has $2\infty - 2m - 2$ finite roots, i.e., $2\infty - 2$ roots in all.

The roots of a differential of the first kind defined on a Hill's surface S constructed from a potential with double eigenvalues can be counted in almost the same way. There is one important modification: in the presence of double eigenvalues S has "fewer" handles, so the descending sequence of spaces $I^{\infty-n}$ must be changed.

Let λ_n^\times , $1 \leq n \leq m \leq \infty$, be the double eigenvalues. Now, the n th space in the chain is by definition all entire functions ϕ for which $\prod_{n=1}^m (1 - \lambda/\lambda_n^\times) \times \phi$ is in $I^{\infty-n}$. Functions ϕ with $(\prod_{n=1}^m 1 - \lambda/\lambda_n^\times) \times \phi$ in $I^\infty \setminus I^{\infty-1}$ require one point on each handle of S for their interpolation, and so on. When all but $2g + 1 < \infty$ of the eigenvalues are double the modified I^∞ is the g -dimensional space of polynomials over \mathbb{C} of degree $g - 1$. This is just as it should be since the genus of S is now g . It seems that the appropriate I^∞ should be designated as the "genus" of S .

The last topic is differentials with infinitely hard roots at ∞ . In §4, we show that a differential $d\Phi \in K$ on a Hill's surface for a real analytic potential has a root of infinite order at ∞ if and only if ϕ is identically zero. This is not

true in general: *the statement is false for potentials which vanish on an interval.* This may be elucidated by means of a remarkable connection between the local vector fields on M and the coefficients

$$c_n(\phi) = 2 \sum_{i>1} (\phi(\lambda_{2i-1})/y_2(1, \lambda_{2i-1}))\lambda_{2i-1}^n \quad (n > 0).$$

Regard c_n as an element of K^\dagger and let $\psi \in K$ be defined by $\psi(\lambda) = (d/d\xi)(\partial\Delta(\lambda)/\partial q(\xi))$ for fixed $0 < \xi < 1$. Then

$$\begin{aligned} \mathbf{V}_1 q &= q'(\xi) = 2c_0(\psi), \\ \mathbf{V}_2 q &= 3q(\xi)q'(\xi) - \frac{1}{2}q'''(\xi) \\ &= 4c_1(\psi) + 2\left(\lambda_0 + \sum_{n>1} l_n\right)c_0(\psi), \end{aligned}$$

and in general, the n th local field \mathbf{V}_n is a linear combination of c_j ($0 < j < n$) so construed. McKean-Trubowitz [1976: 196] proved that the local fields span the tangent space at any point of M if and only if the c_n 's span K^\dagger , which is the same as to say that nontrivial functions $\phi \in K$ with infinitely hard roots at ∞ do not exist. But if $q \in M$ vanishes on an interval so will all the $\mathbf{V}_n q$'s, and these cannot span the tangent space to M at q as the latter contains, e.g., $(d/d\xi)(\partial\Delta(\lambda_0)/\partial q(\xi)) = -\Delta'(\lambda_0)[f_0^2(\xi)]'$ which vanishes only twice.

The theta function. The Riemann theta function for a hyperelliptic surface of genus $g < \infty$ with real branch points is an entire function θ of $z \in \mathbb{C}^g$ defined by

$$\Theta(z) = \sum_{n \in \mathbb{Z}^g} e^{2\pi\sqrt{-1} n \cdot z} e^{\pi\sqrt{-1} Q[n]},$$

in which Q denotes the quadratic form based upon the Riemann matrix of the surface and the sum is over the character group of the real torus R^g/Z^g comprising the exponentials $\exp 2\pi\sqrt{-1} n \cdot x$ ($n \in \mathbb{Z}^g$). In the present case, the *finite* sums $\phi = \sum n_i l_i$ fill out the lattice dual to L_K , and the corresponding exponentials $\exp 2\pi\sqrt{-1} x(\phi)$ comprise the dual group of the real torus $\mathfrak{S} = K^\dagger/L_K$. Now, by analogy with the classical case, the theta function, ϑ , for the Hill's surface S is defined for $z = x + \sqrt{-1} y \in K^\dagger + \sqrt{-1} H^\dagger$ by the formula

$$\vartheta(z) = \sum e^{2\pi\sqrt{-1} z(\phi)} e^{-(\pi/2)H[\phi]},$$

the summation being over the dual lattice described above. The fundamental estimate

$$\sum |e^{2\pi\sqrt{-1} [x(\phi)+\sqrt{-1} y(\phi)]-\pi H[\phi]/2}| \leq ae^{bHy}$$

for $x + \sqrt{-1} y \in K^\dagger + \sqrt{-1} H^\dagger$ justifies this definition, and it is easy to check that ϑ satisfies the customary identities

$$\begin{aligned} \vartheta(z + A_n) &= \vartheta(z), \\ \vartheta(z + B_n) &= e^{-2\pi\sqrt{-1} [z(1_n)+(1/2)B_n(1_n)]} \vartheta(z), \end{aligned}$$

completing the analogy.

Let $z_i = z(1_i)$ ($i \geq 1$) and observe that

$$\begin{aligned} H[\phi] &= H\left[\sum n_i 1_i\right] = -2\sqrt{-1} \sum_{n>1} A_n\left(\sum n_i 1_i\right) B_n\left(\sum n_j 1_j\right) \\ &= -2\sqrt{-1} \sum_{i,j} n_i B_i(1_j) n_j \\ &\equiv -2\sqrt{-1} \sum_{i,j} n_i Q_{ij} n_j = -2\sqrt{-1} Q[n], \end{aligned}$$

Q being the Riemann matrix for S . With this notation, φ assumes the classical form

$$\varphi(z) = \sum e^{2\pi\sqrt{-1}nz} e^{\pi\sqrt{-1}Q[n]},$$

in which the summation is over all tame points $n \in Z^\infty$. We now outline the basic properties of φ which appear in §7.

First of all, φ is continuous on $K^\dagger + \sqrt{-1}H^\dagger$, smooth on $J^\dagger + \sqrt{-1}H^\dagger$, and any translate of φ by a point in K^\dagger is analytic on $H^\dagger + \sqrt{-1}H^\dagger$, i.e., if $x \in K^\dagger$, then $\varphi(x+z)$ is an analytic function of $z \in H^\dagger + \sqrt{-1}H^\dagger$. The classical theta function for a hyperelliptic surface with real branch points is strictly positive on the real part of the Jacobian variety and so it is here: φ is *strictly positive* on K^\dagger . This is important for the variant of the Riemann vanishing theorem of §8.

There is also an analogue of the Jacobi transformation. To motivate it consider an elliptic curve for a Weierstrass p -function with periods 1 and $\sqrt{-1}t$, $t > 0$. Here,

$$\theta(z) = \sum_{n \in Z} e^{2\pi\sqrt{-1}nz} e^{-\pi n^2},$$

and the Jacobi identity states that

$$\theta(x) = \frac{1}{\sqrt{t}} \sum_{n \in Z} e^{-\pi(x-n)^2/t}.$$

Let $dP(x) = t^{-1/2}e^{-\pi x^2/t} dx$ be the Gaussian measure on R with variance $t/2\pi = (1/2\pi\sqrt{-1}) \times$ (the Riemann matrix for the elliptic curve). Then the Jacobi identity may be restated as

$$\int_E \theta(x) dx = \frac{1}{\sqrt{t}} \int_E \sum_n e^{-\pi(x-n)^2/t} dx = \sum_n P(E+n)$$

for $E \subset [0, 1)$, the latter being regarded as the real part of the Jacobian variety. In the present context, let P be the Gaussian probability measure on R^∞ with covariance matrix

$$Ex_i x_j = \frac{1}{2\pi\sqrt{-1}} B_i(1_j) = \frac{1}{2\pi\sqrt{-1}} Q_{ij},$$

let $0 < x'_i < 1$ be congruent to $x_i \pmod{1}$, and let $I(x)$ be the measurable map from $x = (x_1, x_2, \dots) \in R^\infty$ to the torus $K^\dagger/L_K = \mathfrak{S}$ defined by $I(x) = \sum_{n>1} x'_n A_n$. Then the Jacobi transformation states that

$$\int_E \varpi \, d(\text{Haar measure of } \mathfrak{S}) = P(I^{-1}E),$$

where E is any measurable subset of \mathfrak{S} .

The vanishing theorem and the Baker-Its-Matveev formula. Let \mathfrak{p} be any point on S and $\mathfrak{p}_1, \mathfrak{p}_2, \dots$, a sequence of points in real position. The functional¹¹ $x_{\mathfrak{p}}(\phi) = \int_{-\infty}^{\phi} d\Phi + \int_0^{\mathfrak{p}} d\Phi$ belongs to $K^{\dagger} + \sqrt{-1} H^{\dagger}$ and the Abelian sum $\sum_{n \geq 1} x_n(\phi) = \int_{\mathfrak{p}_n}^{\phi} d\Phi$ ($n \geq 1$) is in K^{\dagger} , with the restriction on the winding numbers explained in §1. Now, following Riemann, we consider the function $f(\mathfrak{p}) = \varpi(x_{\mathfrak{p}} - \sum_{n \geq 1} x_n)$. f is multivalued and analytic on S , changing by an exponential multiplicative factor when continued around a cycle. Thus, its zeros are well defined; moreover, it is not identically zero since $f(-\infty) = \varpi(-\sum_{n \geq 1} x_n)$ is positive. The vanishing theorem states that f vanishes simply at \mathfrak{p}_n ($n \geq 1$) and no place else, in perfect analogy to the classical theorem of Riemann. In §8, the present vanishing theorem is used to show that the map from points \mathfrak{p} on S to functionals $x_{\mathfrak{p}}(\phi) = \int_0^{\mathfrak{p}} d\Phi$ in $K^{\dagger} + \sqrt{-1} H^{\dagger}$, mod periods of that class, is an embedding of S into the full Jacobian variety, i.e., $K^{\dagger} + \sqrt{-1} H^{\dagger}$ mod periods of that class.

Now let $x = \sum_{n \geq 1} x_n = \sum_{n \geq 1} \int_{\mathfrak{p}_n}^{\mathfrak{p}_0}$, as above. Then Baker's formula¹² adapted to infinite genus is

$$0 = \frac{\varpi'(x)}{\varpi(x)} + \sum_{n \geq 1} \int_{\mathfrak{p}_n}^{\mathfrak{p}_0} \frac{\Delta'(\lambda)}{2R(\lambda)} \, d\lambda,$$

where the prime signifies differentiation in the direction $2c_0 \in K^{\dagger}$, corresponding to the infinitesimal translation $\mathbf{V}_1 q = q'$. Its-Matveev [1975] discovered a simple way to invert the Jacobi map for finite genus Hill's surfaces using the theta-function. Their formula is derived from Baker's by one more differentiation. Let $x \in \mathfrak{S}$ and let $q \in M$ be its image under the exponential map. Then the infinite genus Its-Matveev formula is

$$q(\xi) = -2 \frac{d^2}{d\xi^2} \log \varpi(x + 2\xi c_0), \quad 0 \leq \xi < 1.$$

In §§8 and 9, the vanishing theorem and Baker's formula are proven by approximating the surface S by finite genus hyperelliptic surfaces. If a good estimate of ϖ at ∞ were available, this approximation would be unnecessary; see Remark 1, §8.

Finally, in §10, we prove the easy half of Abel's theorem in this infinite genus context.

3. Square summable differentials of the first kind. Let $\mathfrak{p} = (\lambda, R(\lambda))$ be a general point of the Riemann surface S , let the integral function ϕ be real on the line, and let $d\Phi(\mathfrak{p})$ be the corresponding differential of the first kind:

$$d\Phi(\mathfrak{p}) = \frac{\phi(\lambda)}{R(\lambda)} \, d\lambda.$$

¹¹ \mathfrak{p}_0 is the point $(\lambda_0, 0)$ on S .

¹²Baker [1897: 232].

A real Hilbert space H of such differentials is defined by the requirement¹³

$$\sqrt{-1} \int_S d\Phi \wedge \overline{d\Phi} = 4 \int_C |\phi/R|^2 d \text{ area} = H[\phi] < \infty.$$

It is handy to speak of $d\Phi$ or ϕ as belonging to H , indifferently, meaning the same thing. The purpose of this article is to prepare for future use some elementary facts about H . Notice, first, that $|R(re^{\sqrt{-1}\theta})| \leq e^{\sqrt{r}}$ implies

$$\begin{aligned} |\phi(\lambda)|^2 &= \frac{1}{\pi^2} \left| \int_{|\omega-\lambda| < 1} \phi(\omega) d \text{ area} \right|^2 \\ &< \frac{1}{\pi^2} H[\phi] \int_{|\omega-\lambda| < 1} |R(\omega)|^2 d \text{ area} \\ &< H[\phi] e^{2\sqrt{r}}, \end{aligned}$$

so that $\phi \in H$ is of order $\leq 1/2$ and type ≤ 1 . The result is confirmed and improved in

THEOREM 1. $H \subset I^{1/2}$, i.e., every function $\phi \in H$ is of order $\leq 1/2$ and type ≤ 1 with $\int_0^\infty |\phi(\lambda)|^2 \lambda^{1/2} d\lambda < \infty$.

LEMMA 1. Let $h(\omega)$ be analytic in the upper half-plane $\omega = a + \sqrt{-1} b$, $b > 0$, and let

$$\int_{-\infty}^\infty da \int_0^\infty db |h(\omega)|^2 < \infty.$$

Then $h(\omega) = \int_0^\infty e^{\sqrt{-1}\omega x} \hat{h}(x) dx$ with $\int_0^\infty |\hat{h}(x)|^2 dx/x < \infty$.

PROOF OF THE LEMMA. The proof may be modeled on Dym-McKean [1972: 162]. The main task is to prove that

$$\|h\|_b^2 = \int_{-\infty}^\infty |h(a + \sqrt{-1} b)|^2 da < \infty$$

is a decreasing function of $b > 0$ by expressing $h(\omega)$ for fixed $\omega = a + \sqrt{-1} b$ as the integral of $[2\pi\sqrt{-1}(\omega' - \omega)]^{-1} h(\omega')$ over a pair of horizontal lines, one above and one below ω . The conclusion is that $\|h\|_b$ is over-estimated by the sum of its values on the upper and lower lines. The contribution from the upper line is eliminated by noting that $\iint da db |h(\omega)|^2 < \infty$ entails $\lim_{b \uparrow \infty} \|h\|_b = 0$. This implies that $h(\omega)$ is of the stated form.

PROOF OF THE THEOREM. Let $\phi \in H$. Then $h(\omega) = \omega\phi(\omega^2)/R(\omega^2)$ satisfies the conditions of Lemma 1, whence

$$\omega\phi(\omega^2) = R(\omega^2) \int_0^\infty e^{\sqrt{-1}\omega x} \hat{h}(x) dx$$

with $\int_0^\infty |\hat{h}|^2 dx/x < \infty$. Now $R^2(\omega^2)$ is an integral function of exponential type. Besides, it is bounded on the real line and over-estimated by $e^{2|\omega|}$ on the imaginary half-line $\omega = \sqrt{-1} b$, $b > 0$. Thus, $e^{2\sqrt{-1}\omega} R^2(\omega^2)$ is bounded in the upper half-plane, by the Phragmén-Lindelöf principle, and

¹³ $\overline{d\Phi}$ is the complex conjugate of $d\Phi$. The factor in front of the second integral is 4 not 2 because S is a double covering of C .

$$e^{\sqrt{-1}\omega}k(\omega) = O(1) \times \int_0^\infty e^{\sqrt{-1}\omega x}\hat{h}(x) dx \tag{a}$$

with $k(\omega) = \omega\phi(\omega^2)$. $\|k\|_b^2$ may now be over-estimated by a multiple of

$$e^{4b} \int_0^\infty e^{-2bx}|\hat{h}(x)|^2 dx < \infty$$

for $b > 0$, and a similar estimate prevails for $b < 0$; in particular,

$$\int_{-1}^1 \|k\|_b^2 db < \infty,$$

and it is easy to deduce

$$\|k\|_0^2 = \frac{1}{2} \int_0^\infty |\phi(\lambda)|^2 \lambda^{1/2} d\lambda < \infty \tag{b}$$

by the methods employed in the proof of Lemma 1. The rest of the proof is easy: (a) implies that $e^{\sqrt{-1}\omega}k(\omega)$ fulfills the conditions of Lemma 1. Thus,

$$e^{\sqrt{-1}\omega}k(\omega) = \int_0^\infty e^{\sqrt{-1}\omega x}k^+(x) dx \quad \text{with} \quad \int_0^\infty |k^+|^2 \frac{dx}{x} < \infty$$

and even $\int_0^\infty |k^+|^2 dx < \infty$, by (b). The same kind of formula can be obtained for $e^{-\sqrt{-1}\omega}k(\omega)$ in the lower half-plane, the upshot being that in the whole plane,

$$k(\omega) = \int_{-1}^1 e^{\sqrt{-1}\omega x}\hat{k}(x) dx \quad \text{with} \quad \int_{-1}^1 |\hat{k}|^2 dx < \infty,$$

or, what is the same by the oddness of $k(\omega) = \omega\phi(\omega^2)$,

$$\phi(\lambda) = 2 \int_0^1 \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \hat{k}(x) dx.$$

The fact that ϕ is of order $< 1/2$ and type < 1 is now plain.

LEMMA 2.

$$\frac{\pi}{4} H[\phi] = \int_0^\infty b^{-2} db \int_{-\infty}^\infty da \left| \int_a^{a+b} d\Phi \right|^2.$$

PROOF. The function $h(\omega) = \phi(\omega)/R(\omega)$ fulfills the conditions of Lemma 1 and so can be expressed as

$$h(\omega) = \int_0^\infty e^{\sqrt{-1}\omega x}\hat{h}(x) dx \quad \text{with} \quad \int_0^\infty |\hat{h}(x)|^2 \frac{dx}{x} < \infty.$$

By the reality of ϕ^2/R^2 on the line,

$$\begin{aligned} H[\phi] &= 8 \int_{-\infty}^\infty da \int_0^\infty db |h(\omega)|^2 \\ &= 16\pi \int_0^\infty db \int_0^\infty e^{-2bx} |\hat{h}(x)|^2 dx = 8\pi \int_0^\infty |\hat{h}(x)|^2 \frac{dx}{x}. \end{aligned}$$

Now begin from the other end:

$$\int_a^{a+b} d\Phi = \int_a^{a+b} h(\omega) d\omega = \int_0^\infty e^{\sqrt{-1}ax} \frac{e^{\sqrt{-1}bx} - 1}{\sqrt{-1}x} \hat{h}(x) dx,$$

as is easily verified by performing the integration a little way up in the half plane and coming down with the help of

$$\int_0^\infty \left| \frac{e^{\sqrt{-1}bx} - 1}{x} \right|^2 |\hat{h}(x)|^2 dx < \infty.$$

It is immediate that

$$\begin{aligned} & \int_0^\infty b^{-2} db \int_{-\infty}^\infty da \left| \int_a^{a+b} d\Phi \right|^2 \\ &= \int_0^\infty b^{-2} db 2\pi \int_0^\infty \left| \frac{e^{\sqrt{-1}bx} - 1}{x} \right|^2 |\hat{h}(x)|^2 dx, \end{aligned}$$

and the stated formula follows by evaluation of

$$\int_0^\infty \left| \frac{e^{\sqrt{-1}bx} - 1}{x} \right|^2 \frac{db}{b^2} = \frac{\pi}{x}$$

and comparison with the previous formula for $H[\phi]$.

USAGE. $L[\phi] = \sum_{i>1} |A_i(\phi)|^2 \log(1/l_i)$ with $l_i = \lambda_{2i} - \lambda_{2i-1}$ ($i \geq 1$); for simplicity it is assumed that $l_i < 1$ ($i \geq 1$) throughout.

THEOREM 2. $L[\phi] < \infty$ if $\phi \in H$.

PROOF. $\phi(\lambda)$ is real on the line while $R(\lambda)$ is real if $\lambda_{2i-1} \leq \lambda \leq \lambda_{2i}$ ($i \geq 1$) and imaginary otherwise, so by Lemma 2,

$$\begin{aligned} \frac{\pi}{4} H[\phi] &\geq \sum_{i>1} \iint b^{-2} da db \left| \int_a^{a+b} d\Phi \right|^2 \\ &\geq \sum_{i>1} \iint b^{-2} da db \times |A_i(\phi)|^2, \end{aligned}$$

in which the double integral in the i th summand is taken over the part of the ab -plane described by $\lambda_{2i-2} \leq a \leq \lambda_{2i-1}$ and $\lambda_{2i} \leq a + b \leq \lambda_{2i+1}$. The proof is finished by evaluation of

$$\int_{\lambda_{2i-2}}^{\lambda_{2i-1}} da \int_{\lambda_{2i}-a}^{\lambda_{2i+1}-a} b^{-2} db = \log \left| \frac{\lambda_{2i+1} - \lambda_{2i-1}}{\lambda_{2i} - \lambda_{2i-1}} \frac{\lambda_{2i} - \lambda_{2i-2}}{\lambda_{2i+1} - \lambda_{2i-2}} \right|$$

and the estimate $\lambda_{2i}, \lambda_{2i-1} = i^2\pi^2 + O(1/i^2)$ ($i \uparrow \infty$), leading to

$$\frac{\pi}{4} H[\phi] \geq \sum_{i>1} |A_i(\phi)|^2 \log \frac{i[\pi^2 + o(1)]}{\lambda_{2i} - \lambda_{2i-1}}.$$

THEOREM 3. If $\phi \in I^{3/2}$ and if $L[\phi] < \infty$, then $\phi \in H$, also; in particular, $H \supset K$.

PROOF. $\phi \in I^{3/2}$ may be interpolated off either λ_{2i-1} ($i \geq 1$) or λ_{2j} ($j \geq 1$),¹⁴ i.e.,

¹⁴McKean-Trubowitz [1976: 174].

$$\begin{aligned} \phi(\lambda) &= \sum_{i>1} \frac{\phi(\lambda_{2i-1})}{\lambda - \lambda_{2i-1}} \frac{y_2(1, \lambda)}{y_2'(1, \lambda_{2i-1})} \\ &= \sum_{j>1} \frac{\phi(\lambda_{2j})}{\lambda - \lambda_{2j}} \frac{y_1'(1, \lambda)}{\lambda - \lambda_0} \frac{\lambda_{2j} - \lambda_0}{y_1'(1, \lambda_{2j})} \end{aligned}$$

in which

$$y_2(1, \lambda) = \prod_{i>1} (i\pi)^{-2} (\lambda_{2i-1} - \lambda)$$

and

$$y_1'(1, \lambda) = (\lambda_0 - \lambda) \prod_{j>1} (j\pi)^{-2} (\lambda_{2j} - \lambda)$$

are computed for the origin of M , namely that q whose tied spectrum is $\mu_n = \lambda_{2n-1}$ ($n \geq 1$). The product of these two sums is substituted into the integral for $H[\phi]$ and $R^2 = y_1' y_2$ is used to obtain

$$\begin{aligned} H[\phi] &= 4 \int_C |\phi/R|^2 d \text{ area} \\ &= 4 \sum_{i,j>1} |\phi(\lambda_{2i-1})\phi(\lambda_{2j})| \times |\lambda_{2j} - \lambda_0| \\ &\quad \times |y_2'(1, \lambda_{2i-1})y_1'(1, \lambda_{2j})|^{-1} \\ &\quad \times \int_C \frac{d \text{ area}}{|(\lambda - \lambda_0)(\lambda - \lambda_{2i-1})(\lambda - \lambda_{2j})|}. \end{aligned}$$

Now $\lambda_{2j} \sim j^2\pi^2$ ($j \uparrow \infty$), while $|y_2'(1, \lambda_{2i-1})|$ is comparable to i^{-2} and $y_1'(1, \lambda_{2j})$ is comparable to 1, with the result that $H[\phi]$ is controlled by

$$\sum_{i,j>1} |\phi(\lambda_{2i-1})\phi(\lambda_{2j})| i^2 j^2 \int_C \frac{d \text{ area}}{|(\lambda - \lambda_0)(\lambda - \lambda_{2i-1})(\lambda - \lambda_{2j})|}.$$

The integral is of the form¹⁵

$$\begin{aligned} &\int_C \frac{d \text{ area}}{|(\lambda - e_1)(\lambda - e_2)(\lambda - e_3)|} \\ &= \int_C \frac{1}{|(p - e_1)(p - e_2)(p - e_3)|} \frac{\sqrt{-1}}{2} dp \wedge \overline{dp} \\ &= 2 \int_F \frac{|p'|^2}{|4(p - e_1)(p - e_2)(p - e_3)|} d \text{ area} \\ &= 2 \times \text{area } F \\ &= -2\sqrt{-1} \times \text{product of the periods of the } p\text{-function} \\ &= 2 \int_{e_2}^{e_1} \frac{dp}{\sqrt{(e_1 - p)(p - e_2)(p - e_3)}} \times \int_{e_3}^{e_2} \frac{dp}{\sqrt{(e_1 - p)(e_2 - p)(p - e_3)}}. \end{aligned}$$

¹⁵ F is the fundamental cell for the p -function with real values e_1, e_2, e_3 at half-periods; an extra factor 1/2 appears in line 3 because p is of degree 2. Notice that the reality of e_1, e_2, e_3 makes the complex period of p pure imaginary; this is used in line 4 of the computation.

This evaluation is used to estimate the integral on the diagonal [$i = j$] and off the diagonal [$i \neq j$], separately: *on the diagonal*, $i = j$, and with $\lambda_0 = a$, $\lambda_{2i-1} = b$, $\lambda_{2i} = c$, the integral is seen to be comparable to¹⁶

$$i^{-1} \int_b^c \frac{dp}{\sqrt{(c-p)(b-p)}} \times \int_a^b \frac{dp}{\sqrt{(p-a)(b-p)(c-p)}} \\ = i^{-1}\pi \times O\left[i^{-1}\log\frac{b}{c-b}\right] = O[i^{-2}\log(1/l_i)].$$

Off-diagonal, a simpler estimate prevails, the integral being comparable to¹⁷

$$[\min(i, j)]^{-1} \times \int_a^b \frac{dp}{\sqrt{(p-a)(b-p)(c-p)}}$$

and so to

$$[\min(i, j)]^{-1} \times |i^2 - j^2|^{-1/2}.$$

The upshot is that $H[\phi]$ is controlled by

$$\sum_{i > 1} |\phi(\lambda_{2i-1})\phi(\lambda_{2i})|i^2\log(1/l_i) \\ + \sum_{\substack{1 < i, j \\ i \neq j}} |\phi(\lambda_{2i-1})\phi(\lambda_{2j})|i^2j^2[\min(i, j)]^{-1}|i^2 - j^2|^{-1/2}.$$

The off-diagonal sum is easy to deal with: $\phi \in I^{3/2}$, so $\sum|\phi(\lambda_{2i-1})|i^4 < \infty$, $\sum|\phi(\lambda_{2j})|j^4 < \infty$, and

$$\left[\sum_{i \neq j} |\phi(\lambda_{2i-1})|i^2|\phi(\lambda_{2j})|j^2[\min(i, j)]^{-1}|i^2 - j^2|^{-1/2} \right]^2$$

is bounded by the product of the former sums and twice

$$\sum_{1 < i < j} i^{-2}(j^2 - i^2)^{-1} = \sum_{i > 1} i^{-2} \sum_{k > 1} (k^2 + 2ik)^{-1} \leq \pi^2/36 < \infty.$$

The on-diagonal sum is just as easy: $\phi \in I^{3/2}$ implies that $\phi'(\lambda) = O(1/\lambda^2)$ for $\lambda \uparrow \infty$, so that if $\lambda = \lambda_{2i-1}$ or λ_{2i} , then the discrepancy between $2\phi(\lambda)\int_{\lambda_{2i-1}}^{\lambda_{2i}} R^{-1}$ and $A_i(\phi) = 2\int_{\lambda_{2i-1}}^{\lambda_{2i}} \phi R^{-1}$ is of magnitude $i^{-3}l_i$, the integral $\int_{\lambda_{2i-1}}^{\lambda_{2i}} R^{-1}$ being comparable¹⁸ to i . Thus, $|\phi(\lambda_{2i-1})\phi(\lambda_{2i})|i^2$ may be over-estimated by a multiple of $|A_i(\phi)|^2 + i^{-6}l_i^2$, the upshot being that the on-diagonal sum is controlled by $L[\phi] < \infty$, since l_i is of rapid decay for $i \uparrow \infty$. The proof is finished.

SUMMING UP. Let $\phi \in L$ signify that $L[\phi] < \infty$. Then $I^{1/2} \cap L \supset H \supset I^{3/2} \cap L$; in particular, $L \supset K$, so $H \supset K$. Thus, H lies somewhere between $I^{1/2}$ and K . In case $l_n \geq ae^{-bn^2}$ ($n \geq 1$), you have $I^{3/2} \subset H$ since $L[\phi]$ is now majorized by $\sum|A_n(\phi)|^2n^2$. This condition is satisfied by, e.g., the Kronig-

¹⁶The contributions to the second integral from the vicinity of a and from the vicinity of b are appraised separately.

¹⁷ a, b, c denote $\lambda_0, \lambda_{2i-1}, \lambda_{2i}$ in order of magnitude.

¹⁸McKean-Trubowitz [1976: 199].

Penney model for electrons in a one-dimensional conductor in which q is a periodic Heaviside function and $l_n \sim 1/n$.

AMPLIFICATION 1. The method of Theorem 2 may also be used to verify that if $\phi \in H$, then $\sum_{i \geq 1} |B_i(\phi)|^2 i^{-2l_i} < \infty$.

PROOF. By Lemma 2,

$$\begin{aligned} \frac{\pi}{4} H[\phi] &\geq \int_{-\infty}^0 da \sum_{i \geq 1} \int_{\lambda_{2i-1}-a}^{\lambda_{2i}-a} b^{-2} db \left| \int_a^{a+b} d\Phi \right|^2 \\ &\geq \int_{-\infty}^0 da \sum_{i \geq 1} \int_{\lambda_{2i-1}-a}^{\lambda_{2i}-a} b^{-2} db \left| \frac{1}{2} B_i(\phi) \right|^2 \\ &= \frac{1}{4} \sum_{i \geq 1} |B_i(\phi)|^2 \log(\lambda_{2i}/\lambda_{2i-1}). \end{aligned}$$

The proof is finished by the appraisal

$$\log(\lambda_{2i}/\lambda_{2i-1}) \sim (i\pi)^{-2} l_i \text{ as } i \uparrow \infty.$$

AMPLIFICATION 2. The method of Theorem 3 may be used to verify that if $\phi \in I^{1/2}$, $L[\phi] < \infty$, and $\sum_{i \geq 1} |A_i(\phi)|^2 i \log i < \infty$, then $\phi \in H$.

4. Roots of a differential of the first kind. It is a fact of classical function theory that a differential of the first kind on a Riemann surface of finite genus g has precisely $2g - 2$ roots. The purpose of this article is to confirm this fact for differentials $d\Phi$ of class J on a Hill's surface. Now the genus is infinite, so it is necessary to devise a mode of counting " $2 \times (g = \text{infinity}) - 2$ " roots. It is also necessary to ascribe a multiplicity to the vanishing of $d\Phi$ at ∞ though there is no local coordinate, in the usual sense of the word, at this point.

LEMMA 1. Let $\phi \in J$ and let

$$c_n(\phi) = \sum_{i \geq 1} \frac{\phi(\lambda_{2i-1})}{y_2(1, \lambda_{2i-1})} \lambda_{2i-1}^n \quad (n \geq 0).$$

Then

$$\phi(\lambda) \sim \lambda^{-3/2} \sqrt{-1} R(\lambda) \sum_{n \geq 0} c_n \lambda^{-n}$$

on nice circles, i.e., circles centered at λ_0 of radius $r \uparrow \infty$ with $|r - \lambda_i| \geq 1$ ($i \geq 1$).

PROOF. $\phi \in J$ may be interpolated off λ_{2i-1} ($i \geq 1$) by the familiar recipe:

$$\frac{\phi(\lambda)}{y_2(1, \lambda)} = \sum_{i \geq 1} \frac{\phi(\lambda_{2i-1})}{y_2(1, \lambda_{2i-1})} \frac{1}{\lambda - \lambda_{2i-1}}.$$

The rapid decrease of $\phi(\lambda_{2i-1})$ ($i \uparrow \infty$) is now used to justify expanding in inverse powers of λ , and the proof is finished by noting that, on nice circles,

$$\frac{R^2(\lambda)}{\lambda [y_2(1, \lambda)]^2} = \frac{\lambda_0 - \lambda}{\lambda} \prod_{i \geq 1} \frac{\lambda_{2i} - \lambda}{\lambda_{2i-1} - \lambda} \sim -1.$$

AMPLIFICATION 1. The branch points $(\lambda_i, 0)$ ($i \geq 1$) pile up at the point ∞ , so that S cannot be given a conventional complex structure there. However, if p approaches ∞ via nice circles or, e.g., via the left half-plane, then the branch points are not very visible, and if you always approach ∞ in this way, $\xi = \lambda^{-1/2}$ is a perfectly adequate local parameter, just as in the case of a hyperelliptic surface of finite genus. Now, by Lemma 1, a differential $d\Phi$ of class J has an expansion

$$d\Phi(p) \sim \lambda^{-3/2} \sqrt{-1} \left[c_0 + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \dots \right] d\lambda$$

$$= \xi^3 \sqrt{-1} [c_0 + c_1 \xi^2 + c_2 \xi^4 + \dots] \times -2d\xi/\xi^3$$

on nice circles, so it is natural to speak of $d\Phi$ as having a root of multiplicity $2n$ at $p = \infty$ if $c_j = 0$ ($j < n$) but $c_n \neq 0$. It is quite possible that $n = \infty$; this case will be commented upon below.

The finite roots of $d\Phi$ come in pairs, so if p is a root, the complementary point p' on the opposite sheet of S is too.¹⁹ Thus, it suffices to count the roots of $d\Phi$ on a single sheet, or equivalently, to count the roots of ϕ in the plane. To do this, it is necessary to introduce the class $I^{\infty-n}$ of integral functions of order $\leq 1/2$ and type ≤ 1 with $I^{\infty-n}[\phi] = \int_0^\infty |\phi(\lambda)|^2 \lambda^{2n+1/2} d\lambda < \infty$. Clearly, $I^\infty = I^{1/2} \supset I^{\infty-1} \supset I^{\infty-2} \dots$. Now to interpolate $\phi \in I^\infty = I^{1/2}$ requires a point on every handle of S , e.g., λ_{2i-1} ($i \geq 1$) will do, and no fewer points suffice. The notation $I^{\infty-n}$ is meant to convey that *to interpolate this smaller class, any n points λ_{2i-1} may be left out, but no more*. This is immediate from the remark that $\prod_{i=1}^n (\lambda - \lambda_{2i-1})\phi(\lambda)$ is of class $I^{1/2}$ if and only if $\phi \in I^{\infty-n}$. These notions lead naturally to the provisional statement that *a differential of the first kind has $2\infty - 2n - 2$ finite roots on S if the $\infty - n - 1$ projections of these roots in the plane suffice to interpolate $I^{\infty-n-1}$ but not $I^{\infty-n}$* . Naturally, the projected roots need not belong to the intervals of instability $[\lambda_{2i-1}, \lambda_{2i}]$ ($i \geq 1$) or even be real, so the mode of interpolation needs to be clarified. Fortunately, there is only one natural way. Let μ_i ($i \geq 1$) be the roots of $\phi \in J$. The latter is of order $\leq 1/2$ so $\sum |\mu_j|^{-1} < \infty$. Thus, for any integral function ψ ,

$$\frac{1}{2\pi\sqrt{-1}} \int \frac{\psi(\mu)\phi(\lambda)}{(\lambda - \mu)\phi(\mu)} d\mu = \psi(\lambda) - \sum \psi(\mu_i) \prod_{j \neq i} \frac{\lambda - \mu_j}{\mu_i - \mu_j},$$

the integral being taken about a circle of radius $r \neq |\mu_i - \lambda_0|$ ($i \geq 1$) centered at λ_0 , and the sum over the roots so enclosed. This assumes the simplicity of the roots: for a root of multiplicity $m + 1$, the corresponding summand must be prefaced by $(m!)^{-1} \partial^m / \partial \mu_i^m$. Be that as it may, it is natural to say that ψ can be interpolated off the roots of ϕ if $\int (\psi/\phi)(\lambda - \mu)^{-1} d\mu = o(1)$ as $r \uparrow \infty$ nicely, i.e., via the radii of nice circles.

THEOREM 1. *Let $d\Phi$ be a differential of the first kind, of class J , with a root of multiplicity $2n < \infty$ at ∞ . Then $d\Phi$ has $2\infty - 2n - 2$ finite roots, i.e., $2\infty - 2$ roots in all.*

¹⁹If p is a branch point [$p = p'$], the root is of multiplicity 2.

LEMMA 2. $|R(re^{\sqrt{-1}\theta})| \sim \exp \sqrt{r} |\sin(\theta/2)|$ on nice circles.

PROOF OF THE LEMMA. The estimate $\lambda_{2i-1}, \lambda_{2i} = i^2\pi^2 + O(i^{-2}) (i \uparrow \infty)$ easily leads to the result. Recall that $\int_0^1 q = 0$. Then the product for $R^2(\lambda) = y_2(1, \lambda)y_1'(1, \lambda)$ may be compared to

$$\left[\frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right]^2 = \prod_{i>1} \left(1 - \frac{\lambda}{i^2\pi^2} \right)^2,$$

with the result that

$$\left| \frac{R(\lambda)}{\sin \sqrt{\lambda}} \right|^2 \sim \sum_{i>1} \left| \frac{\lambda - \lambda_{2i-1}}{\lambda - i^2\pi^2} \right| \left| \frac{\lambda - \lambda_{2i}}{\lambda - i^2\pi^2} \right|.$$

The rest is elementary.

PROOF OF THE THEOREM. $|\phi(\lambda)||R(\lambda)|^{-1} = [c + o(1)]r^{-n-3/2}$ on nice circles with nonvanishing c . Now, for $\psi \in I^{\infty-n-1}$,

$$\infty > I^{\infty-n-1}[\psi] = \int_0^\infty |\psi(\lambda)|^2 \lambda^{2n+5/2} d\lambda = 2 \int_{-\infty}^\infty |\psi(\omega^2)|^2 \omega^{4n+6} d\omega,$$

so

$$\psi(\omega^2)\omega^{2n+3} = \int_0^1 \sin \omega x \hat{\psi}(x) dx \quad \text{with} \quad \int_0^1 |\hat{\psi}(x)|^2 dx < \infty,$$

by the Paley-Wiener theorem, $\psi(\omega^2)$ being of exponential type. Thus,

$$\psi(\lambda) = \lambda^{-n-3/2} \int_0^1 \sin \sqrt{\lambda} x \hat{\psi}(x) dx,$$

and

$$\begin{aligned} |\psi(\lambda)| &= |\lambda|^{-n-3/2} \left[\int_0^1 |\sin \sqrt{\lambda} x|^2 dx \int_0^1 |\hat{\psi}(x)|^2 dx \right]^{1/2} \\ &= O[r^{-n-7/4} e^{\sqrt{r} |\sin(\theta/2)|}] \end{aligned}$$

for $\lambda = re^{\sqrt{-1}\theta}$. By Lemma 2, $|R(re^{\sqrt{-1}\theta})| \sim \exp \sqrt{r} |\sin(\theta/2)|$ on nice circles, so $\int (\psi/\phi)(\lambda - \mu)^{-1} d\mu = O(r^{-n-7/4} \times r^{n+3/2}) = O(r^{-1/4})$ for nice r . This confirms that $I^{\infty-n-1}$ can be interpolated. Contrariwise, the function $\psi(\lambda) = y_2(1, \lambda) \prod_{i=1}^{n+1} (\lambda - \lambda_{2i-1})^{-1}$ is of class $I^{\infty-n}$ and cannot be interpolated:

$$\frac{\psi(\lambda)}{\phi(\lambda)} \sim \frac{1}{c} \frac{\lambda^{-n-1} y_2(1, \lambda)}{\lambda^{-n-3/2} R(\lambda)} \sim \frac{1}{c}$$

on nice circles, so $(2\pi\sqrt{-1})^{-1} \int (\psi/\phi)(\lambda - \mu)^{-1} d\mu \sim 1/c \neq 0 (r \uparrow \infty)$. The proof is finished.

The rest of this section is devoted mostly to the case that $d\Phi$ has an infinitely hard root at ∞ and to the geometrical interpretation of $c_j \in J^\dagger$.

THEOREM 2. $d\Phi \in J$ has a root at ∞ of multiplicity $2n < \infty$ if and only if

$\phi \in I^{\infty-n}$. The statement is also valid if $2n = \infty$ with the interpretation that in such a case ϕ belongs to $I^0 = \bigcap_{n \geq 1} I^{\infty-n}$.

PROOF. It suffices to deal with finite n . Let $\phi \in I^{\infty-n}$. Then

$$|\phi(re^{\sqrt{-1}\theta})| = O(r^{-n-3/4}e^{\sqrt{r}|\sin(\theta/2)|}),$$

as in the previous proof, so that, by Lemma 2, $|\phi(\lambda)| |R(\lambda)|^{-1} = O(r^{-n-3/4})$ on nice circles and $d\Phi$ has a root at ∞ of multiplicity $\geq 2n, n + 3/4$ being larger than $n + 1/2$. Now let $d\Phi$ have a root of multiplicity $2n$. Then $c_j(\phi) = 0 (j < n)$, so

$$\phi(\lambda) = \lambda^{-n} \sum_{i \geq 1} \frac{\phi(\lambda_{2i-1})}{y_2(1, \lambda_{2i-1})} \frac{\lambda_{2i-1}^n}{\lambda - \lambda_{2i-1}},$$

the sum representing a function ψ of class $J \subset I^\infty$, by the rapid decrease of $\phi(\lambda_{2i-1}) (i \uparrow \infty)$. Thus, $I^{\infty-n}[\phi] = I^\infty[\psi] < \infty$. The proof is finished.

THEOREM 3. In the real analytic case, $^{20}l_i \leq ae^{-bi} (i \uparrow \infty)$ and differential $d\Phi$ of class K with an infinitely hard root at ∞ vanishes identically.

PROOF. Let $\phi \in K$. Think of $c_j(\phi)$ as the j th moment of a complex mass distribution placing mass $\phi(\lambda_{2i-1}) [y_2(1, \lambda_{2i-1})]^{-1}$ at the point $\lambda_{2i-1} (i \geq 1)$. Now $|\phi(\lambda_{2i-1})| = O(l_i), \lambda_{2i-1} \sim i^2\pi^2$, and $|y_2(1, \lambda_{2i-1})| \geq O(i^{-2})$, so

$$\begin{aligned} \sqrt[2j]{|c_j(\phi)|} &\leq \sqrt[2j]{\sum_{i \geq 1} O(e^{-bi})i^{2j+2}} \\ &= O\left(\sqrt[2j]{(2j+2)!}\right) = O(j) \quad (j \uparrow \infty), \end{aligned}$$

by Stirling's approximation, and $\sum |c_j(\phi)|^{-1/2j} = \infty$. Now the vanishing of $c_j(\phi) (j \geq 1)$ entails the vanishing of $\phi(\lambda_{2i-1}) (i \geq 1)$ by a criterion of Carleman [1926], so ϕ vanishes, too. The proof is finished.

Now let $\mathbf{V}_j (j \geq 1)$ be the hierarchy of tangent vectors to M defined by $\mathbf{V}_1q = q', \mathbf{V}_2q = 3qq' - (1/2)q''$, and, for general $n \geq 2$, by²¹

$$\mathbf{V}_nq = \left(qD + Dq - \frac{1}{2}D^3\right) \frac{\partial H_{n-1}}{\partial q},$$

and recall the association between such $\mathbf{V} \in T(M)$ and $x \in J^\dagger$ explained in the introduction:

$$x(\phi) = \sum_{i \geq 1} \int_{o_i}^{v_i} d\Phi,$$

in which $v_i = (\mu_i, R(\mu_i)) (i \geq 1)$ represents the image of the origin of M under the action of $e^{\mathbf{V}}$.

THEOREM 4. The element $v_j \in J^\dagger$ corresponding to \mathbf{V}_j is a linear combination of $c_i (i < j)$; for example, $v_1 = 2c_0$, while $v_2 = 4c_1 + 2(\lambda_0 + \sum_{n \geq 1} l_n)c_0$.

PROOF. The formula of McKean-Moerbeke [1975: 232] states that

²⁰See Trubowitz [1977: §3].

²¹ $D = '.$

$$V_j \mu_i = 2 \sum_{k=1}^j \frac{\partial H_{k-1}}{\partial q(0)} (2\mu_j)^{j-k} \frac{R(\mu_i)}{y_2(1, \mu_i)},$$

in which $p_i = (\mu_i, R(\mu_i))$ ($i \geq 1$) corresponds to $q \in M$ and $y_2(1, \lambda) = \prod_{i>1} (i^2 \pi^2)^{-1} (\mu_i - \lambda)$. Let p_i be regarded as a function of $0 \leq t \leq 1$ moving under the V_j -flow, starting at $p_i = o_i$. Then, for $\phi \in J$ and any p_n ($n \geq 1$) in real position

$$\begin{aligned} v_j(\phi) &= V_j \sum_{n>1} \int_{o_n}^{p_n} d\Phi = \sum_{n>1} \frac{\phi(\mu_n)}{R(\mu_n)} V_j \mu_n \\ &= \sum_{i=1}^j \frac{\partial H_{i-1}}{\partial q(0)} 2^{j-i+1} \sum_{n>1} \frac{\phi(\mu_n)}{y_2(1, \mu_n)} \mu_n^{j-i} \\ &= \sum_{i=1}^j \frac{\partial H_{i-1}}{\partial q(0)} 2^{j-i+1} c_{j-i}(\phi), \end{aligned}$$

$c_{j-i}(\phi)$ being formed as in Lemma 1 but with interpolation off the μ_n ($n \geq 1$) at hand rather than off $\mu_n = \lambda_{2n-1}$ ($n \geq 1$). Now evaluate at the origin [$\mu_n = \lambda_{2n-1}$; $n \geq 1$]. Then the new $c_{j-i}(\phi)$ agree with the old and the stated result is obtained; in particular, for $j = 1$, $v_1 = 2c_0$ while for $j = 2$, $v_2 = 4c_1 + 2(\lambda_0 + \sum_{n>1} l_n)c_0$; see McKean-Trubowitz [1976: 223] for more information.

AMPLIFICATION 2. The c_j 's ($j \geq 1$) are now seen to have an interesting geometrical interpretation: they correspond precisely to the local flows induced by V_j ($j \geq 1$) on M . McKean-Trubowitz [1976: 196] proved, in a different language, that V_j ($j \geq 1$) span $T(M)$ if and only if c_j ($j \geq 1$) span K^\dagger . This spanning is now seen to be equivalent to the nonexistence of differentials of the first kind, of class K , with infinitely hard roots at ∞ .

5. Period relations. The purpose of this article is to prove that the classical period relations of Riemann²² hold in the present case with the proper technical interpretation. $d\Phi$ is any differential of the first kind. $A_i(\phi)$ is the integral of $d\Phi$ clockwise about the segment $[\lambda_{2i-1}, \lambda_{2i}]$, as before, while $B_i(\phi)$ is the integral of $d\Phi$ counter-clockwise about the segment $[\lambda_0, \lambda_{2i-1}]$. $A_i(\phi)$ is real and $B_i(\phi)$ is imaginary, by the reality of ϕ on the line.

THEOREM 1. If $\phi \in H$, then $H[\phi] = -2\sqrt{-1} \sum_{i>1} A_i(\phi) B_i(\phi)$ with a technical interpretation of the sum, e.g., if e_n denotes the maximum of

$$\left| 4 \int_{|\lambda|<r} \left| \frac{\phi}{R} \right|^2 d \text{area} + 2\sqrt{-1} \sum_{i=1}^n A_i(\phi) B_i(\phi) \right|$$

for $r \in [\lambda_{2n}, \lambda_{2n+1}]$, then $\sum n^{-1} e_n < \infty$; in particular, $\lim_{n \uparrow \infty} e_n = 0$.

REMARK. A more natural interpretation of the sum is given in §6, where a natural duality between the A 's and B 's is explained.

PROOF. Let D be the region depicted in Figure 3, below, comprising a disk centered at λ_0 of radius $r - \lambda_0$ with $\lambda_{2n} < r < \lambda_{2n+1}$, cut along the segment

²²Siegel [1971: 109].

$[\lambda_0, r]$, let B be its boundary, regarding the cut as having two banks, and let C be just the circle.

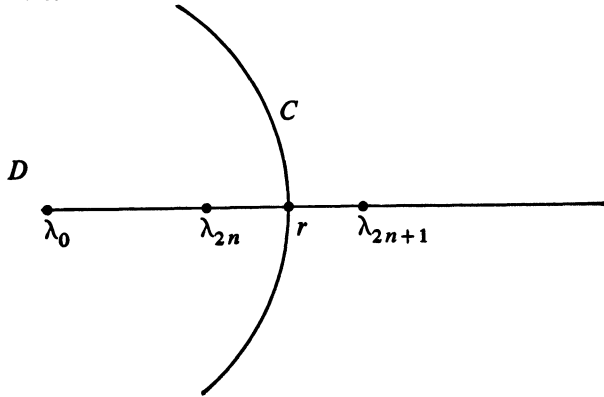


FIGURE 3

Let²³ $\Phi(\lambda) = \int_0^\lambda d\Phi$, the integral being performed without crossing the cut so that Φ is single valued in D , and let $\Phi = \Phi_{\text{real}} + \sqrt{-1} \Phi_{\text{imag}}$. Then, by Stokes' theorem,

$$\begin{aligned} -2\sqrt{-1} \int_D \frac{|\phi|^2}{|R|} d \text{ area} &= \int_D d\Phi \wedge \overline{d\Phi} \\ &= \int_B \Phi \overline{d\Phi} = 2\sqrt{-1} \int_B \Phi_{\text{imag}} d\Phi_{\text{real}} \\ &= 2\sqrt{-1} \sum_{i=1}^n \int_{\lambda_{2i-1}}^{\lambda_{2i}} [\Phi_{\text{imag}}(+)-\Phi_{\text{imag}}(-)] d\Phi_{\text{real}} \\ &\quad + 2\sqrt{-1} \int_C \Phi_{\text{imag}} d\Phi_{\text{real}}, \end{aligned}$$

in which $\Phi(+)$ [$\Phi(-)$] denotes the value of Φ on the upper [lower] bank of the cut, and the fact that $d\Phi$ is purely imaginary in $[\lambda_{2j}, \lambda_{2j+1}]$ ($j < n$) is used in line 4. The sum may be identified as $-\sum_{i=1}^n A_i(\phi)B_i(\phi)$; the final integral may be overestimated by $2L^2(r)$ with $L(r) = \int_C |d\Phi|$, the point being that $\Phi(-r)$ is real so that $\Phi_{\text{imag}}(-r) = 0$ and $|\Phi_{\text{imag}}| \leq L(r)$ on the circle C . The stated period relations result by appraisal of $L(r)$.

LEMMA 1.²⁴ $\int_0^\infty L^2(r)dr/r < \infty$.

PROOF. $\int_0^\infty L^2(r)dr/r$ is compared to $H[\phi]$:

$$\begin{aligned} \int_0^\infty L^2(r) \frac{dr}{r} &= \int_0^\infty \frac{dr}{r} \left[\int_C |\phi| |R|^{-1} r d\theta \right]^2 \\ &< 2\pi \int_0^\infty r dr \int_C d\theta |\phi|^2 |R|^{-2} = \frac{\pi}{2} H[\phi] < \infty. \end{aligned}$$

PROOF OF THEOREM 1 COMPLETED. Let A be the annulus $\lambda_{2n} < r < \lambda_{2n+1}$. Now

²³ λ_0 is the point $(\lambda_0, 0) \in S$.

²⁴Myrberg [1943: 12].

$$\left| 4 \int_D \left| \frac{\phi}{R} \right|^2 d \text{area} + 2\sqrt{-1} \sum_{i=1}^n A_i(\phi) B_i(\phi) \right| \leq 2L^2(r),$$

so

$$e_n \leq 4 \int_A \left| \frac{\phi}{R} \right|^2 d \text{area} + \frac{1}{\lambda_{2n+1} - \lambda_{2n}} \int_{\lambda_{2n}}^{\lambda_{2n+1}} 2L^2(r) dr,$$

and $\sum n^{-1} e_n$ is controlled by

$$\begin{aligned} \sum_{n>1} \frac{1}{(\lambda_{2n+1} - \lambda_{2n})^2} \int_{\lambda_{2n}}^{\lambda_{2n+1}} L^2(r) dr &= \sum_{n>1} O(n^{-2}) \int_{\lambda_{2n}}^{\lambda_{2n+1}} L^2(r) dr \\ &= \int_0^\infty L^2(r) O(r^{-1}) dr < \infty, \end{aligned}$$

by Lemma 1. The proof is finished.

COROLLARY 1.²⁵ *If $\phi \in H$ and if $A_i(\phi) = 0$ ($i \geq 1$), then $\phi = 0$.*

COROLLARY 2. *If $\phi \in H$ and if $B_i(\phi) = 0$ ($i \geq 1$), then $\phi = 0$.*

AMPLIFICATION 1. If $\phi \in I^{3/2} \cap H$, more can be said. Then (a) $\Phi(\infty)$ exists in a technically satisfactory sense, (b) $2\Phi(\infty) = \sum_{i \geq 1} A_i(\phi)$, and (c) $H[\phi] = -2\sqrt{-1} \sum_{i \geq 1} A_i(\phi) B_i(\phi)$ with actual convergence of the sums.

PROOF. The statements (a), (b), (c) all follow from the existence of $\Phi(-\infty)$ and from the fact that $L(r)$ tends to 0 if, e.g., $r \uparrow \infty$ via the midpoints of the segments $[\lambda_{2n}, \lambda_{2n+1}]$; for example, (b) follows from the estimate

$$\left| 2 \int_{-r}^0 d\Phi + \sum_{i=1}^n A_i(\phi) \right| \leq L(r),$$

which is obtained by integration about the boundary of the region depicted in Figure 4, the cuts being regarded as 2-banked, as for Figure 3.

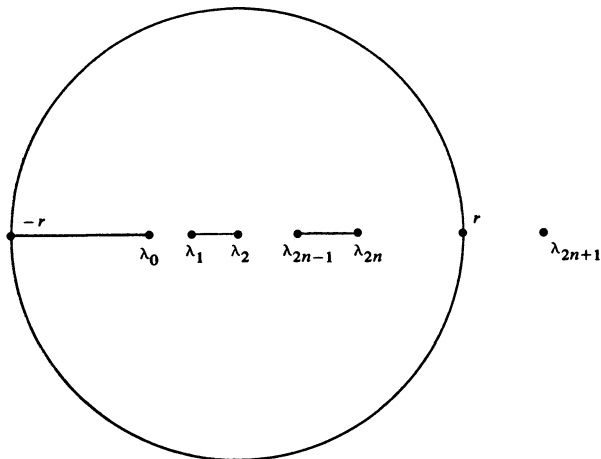


FIGURE 4

²⁵The same is true if $\phi \in I^{1/2} \supset H$ because $\sum |\phi(\mu_i)|^2$ is comparable to $\int_0^\infty |\phi(\lambda)|^2 \lambda^{1/2} d\lambda$. See McKean-Trubowitz [1976: 204] for a similar argument.

Now $h(\omega) = \omega^2\phi(\omega^2)$ is an even integral function of exponential type ≤ 1 . Thus,

$$h(\omega) = \int_0^1 \cos \omega x \hat{h}(x) dx \quad \text{with} \quad \int_0^1 |\hat{h}(x)|^2 dx < \infty,$$

i.e.,

$$\phi(\lambda) = \int_0^1 \frac{\cos \sqrt{\lambda} x}{\lambda} \hat{h}(x) dx,$$

and the existence of $\Phi(-\infty)$ follows from the estimate $|R| \sim e^{\sqrt{|\lambda|}}$ for $\lambda \downarrow -\infty$: $\int_{-\infty}^{-1} |d\Phi|$ is overestimated by a multiple of

$$\begin{aligned} & \int_{-\infty}^{-1} d\lambda e^{-\sqrt{|\lambda|}} \int_0^1 \frac{e^{\sqrt{|\lambda|} x}}{|\lambda|} |\hat{h}(x)| dx \\ &= \int_0^1 |\hat{h}(x)| dx \int_{1-x}^{\infty} e^{-\sqrt{\lambda}} \frac{d\lambda}{\lambda} \\ &= \int_0^1 |\hat{h}(x)| O \left| \log \frac{1}{1-x} \right| dx < \infty. \end{aligned}$$

The proof is finished by a similar estimation of $L(r)$ as $r \uparrow \infty$ via the midpoints of the segments $[\lambda_{2n}, \lambda_{2n+1}]$: on such circles C , $|R| \sim \exp \sqrt{r} |\sin(\theta/2)|$, by Lemma 4.2, so $L(r)$ is bounded above by a multiple of

$$\int_0^\pi r d\theta \int_0^1 \frac{e^{-\sqrt{r} \sin(\theta/2)(1-x)}}{r} |\hat{h}(x)| dx = o(1) \quad (r \uparrow \infty).$$

6. Lattices and quotients. The purpose of this article is to investigate the lattices

$$L_{\text{real}}: \sum_{i \geq 1} n_i A_i, \quad L_{\text{imag}}: \sum_{i \geq 1} n_i B_i,$$

the sums being formed with integral n_i ($i \geq 1$) and viewed as elements of H or of $\sqrt{-1} H$ by means of the natural duality between H and H^\dagger .

LEMMA 1. *H has a natural basis $1_j \in K$ ($j \geq 1$) such that $A_i(1_j) = 1$ if $i = j$ and $A_i(1_j) = 0$ otherwise. The functions 1_j are uniquely determined thereby: $(\phi, 1_j)_H = -2\sqrt{-1} B_j(\phi)$ ($j \geq 1$) in the inner product of H .*

AMPLIFICATION 1. The fact that $A_i(1_j) = 1$ if $i = j$ and vanishes otherwise implies the existence of points $\mu_i \in [\lambda_{2i-1}, \lambda_{2i}]$ ($i \geq 1$) such that $1_j(\mu_i) \times 2 \int_{\lambda_{2j-1}}^{\lambda_{2j}} R^{-1} d\mu = 1$ if $i = j$ and vanishes otherwise. Thus,

$$1_j(\lambda) \times 2 \int_{\lambda_{2j-1}}^{\lambda_{2j}} R^{-1} d\mu = \prod_{i \neq j} \frac{\lambda - \mu_j}{\mu_i - \mu_j},$$

by interpolation. The differentials $d\Phi_j = 1_j R^{-1} d\lambda$ ($j \geq 1$) comprise a normalized basis of the differentials of the first kind.

PROOF. $B_j \in H^\dagger$. Pick $1_j \in H$ so that $(\phi, 1_j) = -2\sqrt{-1} B_j(\phi)$. Then $A_i(1_j) = 1$ if $i = j$ and $A_i(1_j) = 0$ otherwise, by the period relations of Theorem 5.1. The fact that $1_j \in K$ is plain from this; moreover, the

uniqueness of 1_j is seen from Corollary 5.1. The proof is finished by noting that if $\phi \in H$ is perpendicular to 1_j ($j \geq 1$), then $B_j(\phi) = 0$ ($j \geq 1$) and $\phi = 0$ by Corollary 5.2. Thus, 1_j ($j \geq 1$) spans H ; see Amplification 4, below, for a more satisfactory sense in which 1_j ($j \geq 1$) is a basis of H .

THEOREM 1.²⁶ *The image of H under the 1:1 map $A: \phi \rightarrow a_i = A_i(\phi)$ ($i \geq 1$) is the domain $D(Q^{1/2}) \subset E$ of a positive selfadjoint operator $Q^{1/2}$ in E . Let $Q = (Q^{1/2})^2$. Then Q maps $D(Q^{1/2})$ 1:1 onto a space $D(Q^{-1/2}) \supset E$ isomorphic to the dual of $D(Q^{1/2})$. The latter is the image of H under the 1:1 map $\sqrt{-1} B: \phi \rightarrow b_i = \sqrt{-1} B_i(\phi)$ ($i \geq 1$), and there is a natural pairing between $D(Q^{1/2}) = AH$ and $D(Q^{-1/2}) = \sqrt{-1} BH$ such that $Qa = \sqrt{-1} b$ and²⁷*

$$\begin{aligned} H[\phi] &= \sum_{i,j \geq 1} a_i Q_{ij} a_j = Q[a] \\ &= \sqrt{-1} \sum_{i \geq 1} a_i b_i \\ &= -\frac{1}{4} \sum_{i,j \geq 1} b_i Q_{ij}^{-1} b_j = Q^{-1}[b]. \end{aligned}$$

AMPLIFICATION 2. $(\sqrt{-1}/2)Q$ is the Riemann period matrix of the present theory in view of $[A_i(1_j): i, j \geq 1] =$ the identity and $Q_{ij} = (1_i, 1_j)_H = -2\sqrt{-1} B_i(1_j) = -2\sqrt{-1} B_j(1_i)$.²⁸

PROOF OF THE THEOREM. $AH \subset E$ since $H \subset I^{1/2}$ and $|A(\phi)|^2 = \sum a_i^2$ is comparable to $I^{1/2}[\phi] = \int_0^\infty |\phi(\lambda)|^2 \lambda^{1/2} d\lambda < \infty$ in $I^{1/2}$. Now $H[\phi]$ defines a positive quadratic form on AH which is closed as it stands, i.e., AH is closed relative to the graph distance $|A(\phi)| + \sqrt{H[\phi]}$, the latter being comparable to $\sqrt{H[\phi]}$, itself; moreover, AH is dense in E by Lemma 1. This permits the identification of AH as the domain $D(Q^{1/2})$ of a positive selfadjoint operator $Q^{1/2}$ in E , leading to the formula $H[\phi] = \sum a_i Q_{ij} a_j$ with $Q = (Q^{1/2})^2$ and $Q_{ij} = (1_i, 1_j)$ in the inner product of H . Regard Q as a map of $D(Q^{1/2})$ to its dual $D(Q^{-1/2})$. Now the i th component of $QA(1_j)$ is $Q_{ij} = (1_i, 1_j) = -2\sqrt{-1} B_i(1_j)$, so $-2\sqrt{-1} B$ maps $1_j \in H$ into $D(Q^{-1/2})$. It is required to prove that $\sqrt{-1} B$ maps the whole of H onto $D(Q^{-1/2})$. This is easy. $-2\sqrt{-1} B$ defines an element of the dual space of $AH = D(Q^{1/2})$ by means of the formula $(\phi, 1_j) = -2\sqrt{-1} B_j(\phi)$. Thus, $B(\phi)$ may be identified as an element of $D(Q^{-1/2})$, and every such element arises in this way, H being selfdual. The proof is finished.

AMPLIFICATION 3. Q dominates a multiple of the diagonal matrix $[\log(1/l_i): i \geq 1]$ by the final step in the proof of Theorem 5.2; in particular, $Q_{ii}^{-1} < \text{constant} \times [\log(1/l_i)]^{-1}$, and $\text{sp } Q^{-1} < \infty$ in, e.g., the real analytic case $[l_i < ae^{-bi}]$; compare Theorem 2, below.

²⁶ E is the space of points $x = (x_1, x_2, \dots)$ with $|x|^2 = \sum x_i^2 < \infty$.

²⁷The sums have to be interpreted with caution, e.g., $Q[a] = \sum a_i Q_{ij} a_j$ is short for $|Q^{1/2}a|^2$, alias the inner product $a \cdot Qa$ in the natural pairing.

²⁸Siegel [1971: 111] may be consulted for the classical case.

COROLLARY 1.²⁹ $\sum n_i A_i$ converges in H^\dagger if and only if $n \in D(Q^{-1/2})$, i.e., $Q^{-1}[n] = \sum n_i Q_{ij}^{-1} n_j < \infty$.

COROLLARY 2.²⁹ $\sum n_i B_i$ converges in H^\dagger if and only if $n \in D(Q^{1/2})$, i.e., $Q[n] = \sum n_i Q_{ij} n_j < \infty$; that happens only if n is tame.

PROOF. Q dominates a multiple of the identity, so $Q[n] < \infty$ implies $\sum n_i^2 < \infty$.

COROLLARY 3. Every $x \in H^\dagger$ can be uniquely expressed as $\sum x_i A_i$ with $(x_1, x_2, \dots) \in D(Q^{-1/2})$.

COROLLARY 4. Every $y \in \sqrt{-1} H^\dagger$ can be uniquely expressed as $\sum y_i B_i$ with $(y_1, y_2, \dots) \in D(Q^{1/2})$.

AMPLIFICATION 4. Corollary 4 implies that 1_j ($j \geq 1$) is a basis of H in a more satisfactory sense than indicated before: $(\phi, 1_j) = -2\sqrt{-1} B_j(\phi)$, so, by Corollary 4, every $\phi \in H$ can be uniquely expressed as a sum of 1_j 's with coefficients from $D(Q^{1/2})$.

AMPLIFICATION 5. L_{real} may now be identified with the integral points in $D(Q^{-1/2})$. Note that the factor space H/L_{real} lies inside the real part \mathfrak{S} of the Jacobian variety of S , the point being that an injection into $\mathfrak{S} = K^\dagger/L_K$ of the general coset $x + L_{\text{real}} \in H/L_{\text{real}}$ is provided by the inclusion map $x + L_{\text{real}} \rightarrow x + L_K$. L_{imag} may be similarly identified with the (necessarily tame) integral points in $D(Q^{1/2})$. The factor space $\sqrt{-1} H/L_{\sqrt{-1}H}$, or some variant of it, plays the role of the imaginary part of the Jacobian variety of S .

THEOREM 2.³⁰ H/L_H is compact if $\text{sp } Q^{-1} = \sum Q_{ii}^{-1} < \infty$. $\sqrt{-1} H/L_{\sqrt{-1}H}$ is never compact or even of finite diameter.

PROOF. Let $\text{sp } Q^{-1}$ be finite. Let $x = (x_1, x_2, \dots) \in D(Q^{-1/2})$ and let $n = (n_1, n_2, \dots)$ be the integral point selected by chance according to the rule:³¹

$$\begin{aligned} n_i &= [x_i] \quad \text{with probability } 1 - x_i + [x_i] \\ &= [x_i] + 1 \quad \text{with probability } x_i - [x_i], \end{aligned}$$

independently of n_j ($j \neq i$). Then³² $E(n_i) = x_i$, $E(x_i - n_i)^2 < 1$, and for tame x , n is also tame with probability 1 and

$$EQ^{-1}[x - n] = \sum E(x_i - n_i) Q_{ij}^{-1} (x_j - n_j) \leq \text{sp } Q^{-1} < \infty,$$

leading to the appraisal

$$\min_{n \in D(Q^{-1/2})} Q^{-1}[x - n] \leq \text{sp } Q^{-1},$$

²⁹ n_i ($i > 1$) is integral.

³⁰ H/L_H is provided with the natural distance $d(a, b) = \inf \sqrt{H[a - b - c]}$, the infimum being taken over $c \in L_H$.

³¹ $[x]$ is the integral part of x .

³² E is the mean or expectation.

first for tame x and then³³ for general $x \in D(Q^{-1/2})$. Now fix $d = 1, 2, \dots$ and let p be the projection $x \rightarrow (x_1, \dots, x_d, 0, \dots)$. Then

$$\begin{aligned} \min_{n \in D(Q^{-1/2})} Q^{-1}[(1-p)x - n] &< \min_{n \in D(Q^{-1/2})} Q^{-1}[(1-p)(x - n)] \\ &< \sum_{i>d} Q_{ii}^{-1}, \end{aligned}$$

so that H/L_H can be identified with the cell $[0, 1)^d$ modulo a ball of diameter not more than $\sum_{i>d} Q_{ii}^{-1}$. The compactness of H/L_H is plain from this. Now, for $\sqrt{-1} H/L_{\sqrt{-1}H}$, the relevant quadratic form is Q itself, and by Amplification 3, the diameter of the factor space is underestimated by a multiple of

$$\max_{x \in R^d} \min_{n \in Z^d} \sum_{i=1}^d \log(1/l_i)(x_i - n_i)^2 > \frac{1}{4} \sum_{i=1}^d \log(1/l_i).$$

The proof is finished.

AMPLIFICATION 6. H/L_H was injected into \mathfrak{S} in Amplification 5; it is conjectured that *it is the same as \mathfrak{S} if it is compact*, e.g., if $\text{sp } Q^{-1} < \infty$. To prove this, it would be helpful if the functional $x(\phi) = \sum_{i>1} \int_{\alpha_i}^{\beta_i} d\Phi$ could be extended from K to H for β_i ($i \geq 1$) in real position on S . The appraisal $l_i < ae^{-\beta_i^2}$ suffices, and possibly even $l_i < ae^{-\beta_i}$, but it is unclear how to do it in any generality.

7. The theta function. The theta function is defined for $z \in K^\dagger + \sqrt{-1} H^\dagger$ by the formula

$$\vartheta(z) = \sum e^{2\pi\sqrt{-1} z(\phi) - (\pi/2)H[\phi]},$$

the summation being taken over the class of finite sums $\phi = \sum n_i 1_i$: the so-called *tame* elements of H . Recall that

$$H[\phi] = 4 \int_C \left| \frac{\phi}{R} \right|^2 d \text{ area} = -2\sqrt{-1} \sum n_i Q_{ij} n_j = -2\sqrt{-1} Q[n]$$

and let $z_i = z(1_i)$ ($i \geq 1$); in this language, ϑ takes the more recognizable form

$$\vartheta(z) = \sum_{\text{tame } n} e^{2\pi\sqrt{-1} n \cdot z} e^{\pi\sqrt{-1} Q[n]}.$$

The purpose of this article is to prove that the summation makes sense and to derive for future use the most elementary properties of ϑ .

LEMMA 1. *Let $x + \sqrt{-1} y \in K^\dagger + \sqrt{-1} H^\dagger$. Then*

$$\sum |e^{2\pi\sqrt{-1} [x(\phi) + \sqrt{-1} y(\phi)] - (\pi/2)H[\phi]}| < \text{constant} \times e^{4\pi H[y]}.$$

PROOF. By Theorem 3.2,

$$H[\phi] \geq c \sum_{i>1} n_i^2 \log(1/l_i)$$

³³ $D(Q^{-1/2})$ is the closure of tame points relative to the distance $\sqrt{Q^{-1}[x - y]}$.

with a positive constant c . Let $c = 1$ for simplicity; it makes no difference to the proofs. Also, keep in mind the usage established in §3 that $l_i < 1$ ($i \geq 1$). Now

$$\begin{aligned} |e^{2\pi\sqrt{-1}[x(\phi)+\sqrt{-1}y(\phi)]-(\pi/2)H[\phi]}| &= e^{-2\pi y(\phi)-(\pi/2)H[\phi]} \\ &\leq e^{4\pi H[y]}e^{-(\pi/4)H[\phi]} \\ &\leq e^{4\pi H[y]}\exp(\pi/4)\sum_{i>1}n_i^2\log l_i, \end{aligned}$$

the elementary inequality $2ab - (1/2)b^2 \leq 4a^2 - (1/4)b^2$ being employed in line 2. Thus, the sum is over-estimated by the product of $e^{4\pi H[y]}$ and

$$\begin{aligned} \sum_{\text{tame } n} \prod_{i>1} l_i^{n_i^2/2} &< \prod_{i>1} \sum_{m=-\infty}^{\infty} l_i^{m^2/2} \\ &= \prod_{i>1} [1 + O(l_i^{1/2})] < \infty \end{aligned}$$

since l_i is rapidly decreasing. The proof is finished.

COROLLARY 1. $\omega(z)$ is well defined on $K^\dagger + \sqrt{-1}H^\dagger$.

COROLLARY 2. Let (p_1, p_2, \dots) be in real position on S , as for the Jacobi map, and let p be any point on S . Then the functional

$$x(\phi) = \int_{\infty}^p d\Phi - \sum_{i>1} \int_{o_i}^{p_i} d\Phi = x_p(\phi) - \sum_{i>1} x_i(\phi)$$

belongs to $K^\dagger + \sqrt{-1}H^\dagger$, as may be seen by splitting the integral from ∞ to p into 2 pieces:³⁴ $\int_{\infty}^p = \int_{-\infty}^o + \int_o^p$. The first piece defines an element of $I^{3/2\dagger} \subset K^\dagger$, while the second piece belongs to $H^\dagger + \sqrt{-1}H^\dagger$. Thus, $\omega(x_p - \sum_{i>1}x_i)$ is well defined.

The transformation properties of ω are just as in the classical case. This is the content of

THEOREM 1. Let $z \in K^\dagger + \sqrt{-1}H^\dagger$, let $A_n \in H^\dagger$ be any real period, and let $B_n \in \sqrt{-1}H^\dagger$ be any imaginary period. Then

$$\begin{aligned} \omega(z + A_n) &= \omega(z), \\ \omega(z + B_n) &= e^{-2\pi\sqrt{-1}[z(1_n)+(1/2)B_n(1_n)]} \omega(z); \end{aligned}$$

in particular, by the first rule, ω is periodic in K^\dagger , i.e., it is really a function on $\mathfrak{S} = K^\dagger/L_K$.

PROOF. The first rule is self evident. The second is derived as in the classical case. The sum for ω is rearranged by substituting $\phi + 1_n$ for ϕ ; this is permitted because 1_n is tame. Now,

³⁴ o is the point $(\lambda_\phi, 0) \in S$.

$$\begin{aligned} \mathfrak{O}(z) &= \sum e^{2\pi\sqrt{-1} z(\phi+1_n)} e^{-(\pi/2)H[\phi+1_n]} \\ &= e^{2\pi\sqrt{-1} z(1_n)} e^{-(\pi/2)H[1_n]} \sum e^{2\pi\sqrt{-1} z(\phi)} \\ &\qquad \qquad \qquad \times e^{-(\pi/2)(H[\phi]+2(\phi,1_n))} \\ &= e^{2\pi\sqrt{-1} [z(1_n)+(1/2)B_n(1_n)]} \sum e^{2\pi\sqrt{-1} z(\phi)} \\ &\qquad \qquad \qquad \times e^{2\pi\sqrt{-1} B_n(\phi)} e^{-(\pi/2)H[\phi]} \\ &= e^{2\pi\sqrt{-1} [z(1_n)+(1/2)B_n(1_n)]} \mathfrak{O}(z + B_n). \end{aligned}$$

The identity $(\phi, 1_n) = -2\sqrt{-1} B_n(\phi)$ is used in the third line.

THEOREM 2. \mathfrak{O} is continuous on $K^\dagger + \sqrt{-1} H^\dagger$, smooth on $J^\dagger + \sqrt{-1} H^\dagger$, and any translate on \mathfrak{O} by a point in K^\dagger is analytic³⁵ on the complexification of H^\dagger .

PROOF that \mathfrak{O} is continuous on $K^\dagger + \sqrt{-1} H^\dagger$: $|e^{\sqrt{-1}\xi} - e^{\sqrt{-1}\eta}| < 2|\xi - \eta|^\delta$ for any $0 < \delta < 1$, so for x_1 and $x_2 \in K^\dagger$, and

$$\begin{aligned} |\mathfrak{O}(x_1) - \mathfrak{O}(x_2)| &< \sum |e^{2\pi\sqrt{-1} x_1(\phi)} - e^{2\pi\sqrt{-1} x_2(\phi)}| e^{-(\pi/2)H[\phi]} \\ &< 4\pi \sum_{\text{tame } n} |x_1(\phi) - x_2(\phi)|^\delta e^{\Sigma n^2 \log l}. \end{aligned}$$

Now for $x \in K^\dagger$ and $x_i = x(1_i)$ ($i > 1$)

$$|x(\phi)|^2 < \sum x_i^2 l_i^2 \sum n_i^2 / l_i^2 = K[x] \sum n_i^2 / l_i^2,$$

so the final sum is bounded by the product of $2K[x_1 - x_2]$, raised to the $\delta/2$ power, and

$$\begin{aligned} \sum_{\text{tame } n} \left[\sum_{i>1} n_i^2 / l_i^2 \right]^{\delta/2} &\times \prod_{i>1} l_i^{n_i^2} \\ &< \sum_{\text{tame } n} \prod_{i>1} [1 + n_i^2 / l_i^2]^{\delta/2} l_i^{n_i^2} \\ &< \prod_{i>1} \sum_{n=-\infty}^{\infty} [(1 + n^2 / l_i^2)]^{\delta/2} l_i^{n^2}. \end{aligned}$$

It is required to prove that this quantity can be made finite by proper choice of $0 < \delta < 1$. But for $l < 1$ and $\delta < 1/2$,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (1 + n^2 / l^2)^{\delta/2} l^{n^2} &< 1 + 4 \sum_{n>1} (n/l)^\delta l^{n^2} \\ &= 1 + O(l^{1/2}), \end{aligned}$$

and that does the trick.

PROOF that \mathfrak{O} is smooth on $J^\dagger + \sqrt{-1} H^\dagger$: The differentiability of \mathfrak{O} is confirmed by estimating $\sum |x(\phi)|^p e^{-(\pi/2)H[\phi]}$ for $p \geq 1$ and $x \in J^\dagger$. Now

³⁵A function f defined on a complex Hilbert space is analytic if $f(x + \omega y)$ is an analytic function of $\omega \in \mathbb{C}$.

$x(\phi) = \sum x_i n_i$ with $x_i = x(1_i)$ and $n_i = A_i(\phi)$, so $x(\phi)$ is bounded by a multiple of $\sum i^q n_i^2$ for some $q < \infty$. Thus, the sum in question is bounded by a multiple of

$$\sum_{\text{tame } n} \prod_{i>1} (1 + i^q n_i^2)^p l_i^{n_i^2} < \prod_{i>1} \sum_{n=-\infty}^{\infty} (1 + i^q n^2)^p l_i^{n^2}.$$

The proof is finished by estimating

$$\sum_{n=-\infty}^{\infty} (1 + i^q n^2)^p l_i^{n^2} < 1 + 2^{1+p} \sum_{n>1} i^{pq} n^{2p} l_i^{n^2} = 1 + O(l_i^{1/3}).$$

PROOF that $\phi(x + \cdot)$ is analytic on the complexification of H^\dagger when $x \in K^\dagger$: This is immediate from the estimate of Lemma 1: in fact, for $x \in K^\dagger$ and y from the complexification of H^\dagger ,

$$\sum |e^{2\pi\sqrt{-1} [x(\phi) + \omega y(\phi)] - (\pi/2)H[\phi]}| < \text{constant} \times e^{4\pi|\omega|^2 H[y]}$$

for any complex ω .

COROLLARY 3. $f(p) = \phi(x_p - \sum_{i>1} x_i)$ is a (many-valued) analytic function of $p \in S$.

PROOF. By Lemma 1, it suffices to remark that

$$\left| \int_0^p d\Phi \right| < \text{constant} \times \sqrt{H[\phi]},$$

locally on S . Observe that $\int_0^p d\Phi$ is a weakly analytic map of p into the complexification of H^\dagger and, as such, is strongly analytic.

THEOREM 3. $\phi(x)$ is positive on K^\dagger .

AMPLIFICATION 1. This fact will be important in connection with the Riemann vanishing theorem, a variant of which is proved below.

PROOF.³⁶ Introduce the Gaussian distribution $dP(x)$ on R^∞ specified by³⁷ $E x_i x_j = (2\pi\sqrt{-1})^{-1} Q_{ij}$. Let $0 < x'_i < 1$ be congruent to x_i , modulo 1. Then

$$x = (x_1, x_2, \dots) \rightarrow x' = \sum x'_i A_i$$

defines a measurable mapping of x into $K^\dagger/L_K = \mathfrak{S}$. Now \mathfrak{S} is an ∞ -dimensional torus with characters $\exp 2\pi\sqrt{-1} x(\phi)$, indexed by the tame functions $\phi = \sum n_i 1_i$, and you may compute the ϕ th Fourier coefficient of the distribution dP' of x' as follows:

$$\begin{aligned} E e^{2\pi\sqrt{-1} x(\phi)} &= E e^{2\pi\sqrt{-1} x(\phi)} = E e^{2\pi\sqrt{-1} \sum x_i n_i} \\ &= e^{\pi\sqrt{-1} Q[n]} = e^{-\pi H[\phi] / 2}. \end{aligned}$$

The upshot is that $dP' = \phi(x) dx$, dx being the Haar measure of \mathfrak{S} , from which it is apparent that $\phi \geq 0$ on K^\dagger since $\phi \in C(K^\dagger)$ and the support of dx is dense in \mathfrak{S} . Now let

³⁶S. R. S. Varadhan helped us with this proof.

³⁷ E is the mean or expectation.

$$\varphi_{1/2}(x) = \sum e^{2\pi\sqrt{-1}x(\phi) - \pi H|\phi|/4}.$$

Then $\varphi_{1/2} \in C(K^\dagger)$ just like φ ; also,

$$\varphi(x) = \int_{\mathfrak{S}} \varphi_{1/2}(x-y) \varphi_{1/2}(y) dy$$

and if $\varphi(x)$ vanishes at some point $x_0 \in \mathfrak{S}$, then $\varphi_{1/2}(x_0-y) \varphi_{1/2}(y)$ vanishes identically on \mathfrak{S} . But $\varphi_{1/2}(0) \neq 0$, so $\varphi_{1/2}$ vanishes in the vicinity of x_0 ; it is required to prove that this cannot happen. The distributions $dP_{1/2}$ and $dP'_{1/2} = \varphi_{1/2}(x) dx$ associated with $2^{-1/2}x$ should now be introduced, but the extra notation is burdensome, and it is simpler to finish the proof under the assumption that φ , itself, vanishes on an open set: the argument is just the same. Now under the translation $x \rightarrow x+y$ with $y \in H^\dagger$, $dP(x)$ transforms by the factor³⁸ $e^{2\pi x(y) - (\pi/2)H[y]}$, i.e., $dP(x+y)/dP(x)$, evaluated at $x = x'$ is given by $e^{2\pi x'(y) - (\pi/2)H[y]}$, $x(y)$ being the Gaussian quantity $\sum x_i y_i$ with $y_i = A_i(y)$ ($i > 1$); the latter makes sense because

$$2\pi E \left| \sum_{i < d} x_i y_i \right|^2 = -\sqrt{-1} \sum_{i,j < d} y_i Q_{ij} y_j \leq H[y] < \infty.$$

Let F' be the field of events concerning x which are insensitive to integral translations, i.e., the smallest field over which x' is measurable. Then, by the previous remarks, $dP'(x+y)/dP(x)$, evaluated at $x = x'$, is given by $E[e^{2\pi x(y) - (\pi/2)H[y]} | F']$ the conditional expectation being $< \infty$ and > 0 , with probability 1. Now if dP' vanishes on an open set $U \subset \mathfrak{S}$, then it will also vanish on any translate $U+y$, provided $y \in H^\dagger$. The proof is finished by the remark that a finite number of such translates cover \mathfrak{S} , contradicting $P'(\mathfrak{S}) = 1$; $\mathfrak{S} = K^\dagger/L_K$, being compact, is covered by a finite number of translates with $y \in K^\dagger$. But $H^\dagger \subset K^\dagger$ is dense in the latter, so the necessary translates may be taken from H^\dagger .

AMPLIFICATION 2. The proof of Theorem 3 leads to an amusing variant of the Jacobi transformation of the theta-function in the present setting. Formally,

$$dP(x) = \frac{e^{-\pi Q^{-1}[x]}}{\sqrt{\det Q}} d^\infty x,$$

in which $d^\infty x$ is the (formal) flat measure on R^∞ . Because $dP'(x)$ is $\varphi(x) dx$, this suggests a Jacobi formula:

$$\text{vol}(\mathfrak{S}) \sum_{\text{integral } n} \frac{e^{-Q^{-1}[x+n]}}{\sqrt{\det Q}} = \varphi(x),$$

which is only formal, too, since e.g., $\text{vol}(\mathfrak{S}) = \det Q = \infty$. The proper interpretation of the Jacobi formula is the relation

$$\int_{E'} \varphi(x) dx = P(E),$$

³⁸This is the formula of Cameron-Martin [1944]. It is closely related to Theorem 1.

in which E is the inverse image of the measurable set $E' \subset \mathfrak{S}$ relative to the map $x \rightarrow x'$ of R^∞ into \mathfrak{S} .

AMPLIFICATION 3. An improvement upon Lemma 1 is required below. This states that, for $x \in K^\dagger, y \in H^\dagger$, and $|n| = \sum |n_i|$,

$$\sum_{|n| > m} |e^{2\pi\sqrt{-1}[x(\phi) + \sqrt{-1}y(\phi)] - (\pi/2)H[\phi]}| \leq e^{-m} e^{(4\pi)H[y]}.$$

PROOF. Let $y_i = A_i(y)$ ($i \geq 1$). Then the sum in question is over-estimated, just as in Lemma 1, by the product of $e^{(4\pi)H[y]}$ and

$$\begin{aligned} \sum_{\substack{\text{same } n \\ |n| > m}} \prod_{i > 1} l_i^{n_i} &\leq e^{-m} \prod_{i > 1} \sum_{n=-\infty}^{\infty} e^{|n|} l_i^{n^2/2} \\ &= e^{-m} \prod_{i > 1} [1 + O(l_i^{1/2})] < \infty. \end{aligned}$$

8. The vanishing theorem. Let $x_p(\phi) = \int_\infty^p d\Phi$ and $x_i(\phi) = \int_{o_i}^{p_i} d\Phi$ with p_i ($i \geq 1$) is real position on S , as for the Jacobi map. Then $f(p) = \phi(x_p - \sum_{i > 1} x_i)$ is analytic in p , by Corollary 7.3; also, $f(\infty) = \phi(-\sum_{i > 1} x_i) \neq 0$, by Theorem 7.3, so that $f(p)$ does not vanish identically. The next result is a variant of the Riemann vanishing theorem; see Baker [1897: 296–342] for the classical case employed in the proof and Siegel [1971: 165–172] for general information.

THEOREM 1. $f(p)$ vanishes simply at p_i ($i \geq 1$) and no place else.

REMARK 1. The classical proof³⁹ for surfaces of finite genus could have been followed if a good appraisal of $f(p)$ in the vicinity of $p = \infty$ had been available; as it is, the easiest way is to approximate ϕ by theta-functions of such surfaces and to make $g \uparrow \infty$.

PROOF. Fix the genus $g = 1, 2, \dots$, let $R(\lambda)$ be replaced by

$$R_g(\lambda) = \sqrt{-\prod_{i=0}^{2g} (\lambda - \lambda_i) \pi^{-4g} (g!)^{-4}},$$

the numerical factor being introduced to make R_g tend to R as $g \uparrow \infty$, and let S_g, H_g , and Θ_g be defined in analogy to the case $g = \infty$. Let p_i ($i \geq 1$) be fixed in real position on S , let paths from o_i to p_i be chosen there, as for the Jacobi map, not winding more than once about their respective circles, and let $\int_{o_i}^{p_i}$ ($i = 1, \dots, g$) signify integration along the same paths on S_g . Let $p = (\mu, R(\mu))$ be a general point of S and let the same letter denote the point $(\mu, R_g(\mu))$ of S_g determined by ascribing to R_g the same signature as to R . Note that this is stable for $g \uparrow \infty$, locally in p , i.e., if p is confined to a compact piece of S and if g is sufficiently large, then the signature depends upon p but not upon g . Let a path from ∞ to p be fixed on S and let \int_∞^p signify integration along the same path on S_g , assuming that g is large enough for this to be possible. Now introduce the functional $x_g = \int_\infty^p - \sum_{i=1}^g \int_{o_i}^{p_i}$ acting upon the general differential of the first kind for S_g . This represents a point of the Jacobian variety \mathfrak{S}_g of S_g , and, in this language, the classical vanishing

³⁹Siegel [1971: 165].

theorem states that $f_g(\wp) = \Theta_g(f_\infty^\wp - \sum_{i=1}^g f_{o_i}^\wp)$ vanishes simply at $\wp = \wp_i$ ($i = 1, \dots, g$) and no place else;⁴⁰ it is required to prove that the same statement holds for $g = \infty$. Now $f_g(\wp)$ is analytic in \wp , and if it could be proved that it approximates $f(\wp) = \omega(f_\infty^\wp - \sum_{i>1} f_{o_i}^\wp)$, e.g., locally boundedly, as $g \uparrow \infty$, then the proof could be finished by use of the classical theorem of Hurwitz. Now, on S_g , the general differential of the first kind is of the form $d\Phi = \phi R_g^{-1} d\lambda$ with a polynomial ϕ of degree $< g$, and you may pick a normalized basis $d\Phi_j = 1_j R_g^{-1} d\lambda$ ($j = 1, \dots, g$) of differentials of the first kind such that $2 \int_{\lambda_{2i-1}}^{\lambda_{2i}} d\Phi_j = 1$ if $i = j$ and vanishes otherwise. In this language, the theta function can be expressed as

$$\Theta_g(x) = \sum e^{2\pi\sqrt{-1} x(\phi) - (\pi/2)H_g[\phi]},$$

the sum being taken over $\phi = \sum_{i=1}^g n_i 1_i$, and the appraisal of Amplification 7.3 applies:

$$\sum_{|n|>m} |e^{2\pi\sqrt{-1} x(\phi) - (\pi/2)H_g[\phi]}| < c_1 e^{-m} e^{4\pi H_g[\text{imag } x]}$$

with a constant c_1 independent of g . The moral is that the tail of the sum may be neglected if $H_g[\text{imag } x]$ is controlled and that it suffices for the proof of Theorem 1 to check two points: (a) that the individual summands of $\Theta_g(f_\infty^\wp - \sum_{i=1}^g f_{o_i}^\wp)$ approximate the summands of $\omega(f_\infty^\wp - \sum_{i>1} f_{o_i}^\wp)$, and (b) that $H_g[\text{imag } f_\infty^\wp]$ is bounded, independently of $g \uparrow \infty$, for \wp confined to a compact part of S .

LEMMA 1. Let $d\Phi_j = 1_j R_g^{-1} d\lambda$ ($j = 1, \dots, g$) be the differentials of the first kind introduced above. Then it is possible to make $g \uparrow \infty$ in such a way that 1_j tends to the analogous function for $g = \infty$, locally uniformly, for every $j = 1, 2, \dots$.

PROOF. $2 \int_{\lambda_{2i-1}}^{\lambda_{2i}} d\Phi_j = 1$ if $i = j$ and vanishes otherwise, so it is possible to select points $\mu_i \in [\lambda_{2i-1}, \lambda_{2i}]$ ($i = 1, \dots, g$) such that

$$\begin{aligned} 1_j(\mu_i) \times 2 \int_{\lambda_{2j-1}}^{\lambda_{2j}} R_g^{-1} d\mu &= 1 \quad \text{if } i = j, \\ &= 0 \quad \text{if } i \neq j. \end{aligned}$$

Now $\text{deg } 1_j < g$, so 1_j may be recovered from this information by interpolation,

$$1_j(\lambda) \times 2 \int_{\lambda_{2j-1}}^{\lambda_{2j}} R_g^{-1} d\mu = \prod_{\substack{1 \leq i < g \\ i \neq j}} \frac{\lambda - \mu_i}{\mu_j - \mu_i};$$

just as in Amplification 6.1. The estimate $\lambda_{2i-1}, \lambda_{2i} = i^2\pi^2 + O(i^{-2})$ ($i \uparrow \infty$), combined with the convergence of the integral to $2 \int_{\lambda_{2j-1}}^{\lambda_{2j}} R^{-1} d\lambda$, permits making $g \uparrow \infty$ so as to make 1_j converge locally uniformly in the plane. The limiting differential is denoted by $d\Phi_j^\infty = 1_j^\infty R^{-1} d\lambda$; it is of class K since

⁴⁰Baker [1897: 297]. This variant of the vanishing theorem is peculiar to the hyperelliptic case: the special choice of base points o_i ($i = 1, \dots, g$) avoids the introduction of the Riemann constant $\sum_{i=1}^g f_{o_i}^\infty$.

$1_j^\infty(\lambda)$ is of the form $y_2(1, \lambda)(\lambda - \lambda')^{-1}$, λ' being a root of $y_2(1, \lambda) = 0$, and may be identified by the fact that $2 \int_{\lambda_{2i-1}}^{\lambda_{2i}} d\Phi_j^\infty = 1$ if $i = j$ and vanishes otherwise. The proof is finished.

LEMMA 2. $|1_j(\lambda)| |R_g(\lambda)|^{-1} < c_2 |\lambda|^{-3/2}$ for $-\infty < \lambda < \lambda_0 - 1$, say, with a constant c_2 depending upon j but not upon g ; in particular, if $g \uparrow \infty$ in the manner of Lemma 1, then $\int_\infty^p d\Phi_j$ tends to the analogous quantity for $g = \infty$, locally uniformly in p .

PROOF. The estimation is elementary: for fixed $j > 1$, large positive λ , and μ_i ($i = 1, \dots, g$) as in the proof of Lemma 1,

$$|1_j(\lambda)| |R_g(\lambda)|^{-1} < c_3 |\lambda|^{-3/2} \prod_{i>2} [1 + O(l_i)].$$

The proof is finished by use of the rapid decrease of l_i .

LEMMA 3. $|1_j(\lambda)| |R_g(\lambda)|^{-1} < c_3 i^{-3} l_i [(\lambda - \lambda_{2i-1})(\lambda_{2i} - \lambda)]^{-1/2}$ for $\lambda_{2i-1} < \lambda < \lambda_{2i}$ with a constant c_3 depending upon j but not upon i or g ; in particular, the tail of $\sum_{i=1}^g \int_{\mu_i}^p d\Phi_j$ is negligible, independently of g .

PROOF. The estimation is much as before: just appraise the i th factors of the products for 1_j and R_g^2 , separately.

PROOF OF (a). This is now plain from Lemmas 1, 2, 3; in fact, for tame n and for $g \uparrow \infty$ in the manner of Lemma 1,

$$x_g \left[\sum_{j=1}^g n_j 1_j \right] = \sum_{j=1}^g n_j \left[\int_\infty^p d\Phi_j - \sum_{i=1}^g \int_{\mu_i}^{p_i} d\Phi_j \right]$$

tends to the analogous quantity for $g = \infty$, locally uniformly in p , while $H_g[\sum_{j=1}^g n_j 1_j]$ behaves itself, too, as may easily be seen by writing it out in terms of imaginary periods:

$$- H_g \left[\sum_{j=1}^g n_j 1_j \right] = \sum_{i,j < g} n_i n_j \times 2\sqrt{-1} \int_{\lambda_0}^{\lambda_{2i-1}} d\Phi_j.$$

The proof is finished.

PROOF OF (b). It is required to control $H_g[\text{imag } \int_\infty^p]$ as $g \uparrow \infty$. \int_∞^p differs from \int_0^p by something real, and

$$\left| \int_0^p \sum_{j=1}^g n_j d\Phi_j \right| < \int_0^p |R_g|^{-1} d \text{ length} \times \max_{\text{op}} \left| \sum_{j=1}^g n_j 1_j \right|.$$

Let $\sum_{j=1}^g n_j 1_j = \phi$ and let D be a disc enclosing the projection of the path onto the plane. Then

$$\max_{\text{op}} |\phi|^2 < c_4 \int_D |\phi|^2 d \text{ area} < c_4 \int_D |\phi|^2 |R_g|^{-2} d \text{ area} \times \max_D |R_g|^2 < c_5 H_g[\phi]$$

with a constant c_5 depending upon D but not upon g , the upshot being

⁴¹₀ = $(\lambda_0, 0)$.

$$H_g \left[\text{imag} \int_0^p \right] < c_5 \left[\int_0^p |R_g|^{-1} d \text{length} \right]^2 = c_6,$$

independently of $g \uparrow \infty$. A variant of the vanishing theorem is expressed in

THEOREM 2. $\omega(\sum_{i \geq 1} f_{0_i}^{p_i})$ viewed as a function of p_1 vanishes simply at the points p'_2, p'_3, \dots complementary⁴² to p_2, p_3, \dots ; it has no other roots besides $p_1 = \infty$.

PROOF. $\sum_{i \geq 1} f_{0_i}^{p_i}$ may be rewritten as $f_{\infty}^{p_1} + f_{0_1}^{\infty} - \sum_{i \geq 2} f_{0_i}^{p_i}$; moreover, as it is a question only of the vanishing of ω , $f_{0_1}^{\infty}$ can be replaced by $-f_{0_1}^{\infty}$, the latter differing from the former by a real period. Thus, it suffices to look at $f(p_1) = (f_{\infty}^{p_1} - f_{0_1}^{\infty} - \sum_{i \geq 2} f_{0_i}^{p_i})$. This function vanishes simply at p'_2, p'_3, \dots and no place else on S as may be seen by extending the proof of Theorem 1 to cover the case that one of the fixed points p_i ($i \geq 1$) is removed to ∞ . The fact that $f(p_1)$ also vanishes at $p_1 = \infty$ is seen in the same way.

AMPLIFICATION 1. The vanishing theorem extends in a selfevident way to the case in which a finite number of the points p_i ($i \geq 1$) lie in general position on S .

COROLLARY 1. Let all but a finite number of points p_i ($i \geq 1$) be in real position on S . Then the set of points $x \in K^\dagger + \sqrt{-1} H^\dagger$ expressible as $f_{0_i}^{p_i} - \sum_{j \neq i} f_{0_j}^{p_j}$ is independent of the choice of $i = 1, 2, \dots$. $\omega(x)$ vanishes on this set and conversely: modulo periods, these are the only roots of $\omega(x) = 0$ in $K^\dagger + \sqrt{-1} H^\dagger$ expressible as $x = f_{\infty}^{p_1} + \sum_{i \geq 1} f_{0_i}^{p_i}$ with only a finite number of the points p_i ($i \geq 1$) in general position on S .

COROLLARY 2. In the real analytic case⁴³ the map $p \rightarrow x_p = f_p^{p_i}$ viewed modulo periods of class $K^\dagger + \sqrt{-1} H^\dagger$, is an embedding of S into the associated complex Jacobian variety.⁴⁴

PROOF. $x_p \in K^\dagger + \sqrt{-1} H^\dagger$, and the fact that the map is 1:1 is immediate from the vanishing theorem: if p_i ($i \geq 2$) are in real position on S and if p_1, q_1 are different from these, then the function $f(p)$ of Theorem 1 vanishes at q_1, p_1, p_2, \dots , and that is one root too many. Now $x_p = \sum_{j \geq 1} x_p(1_j) A_j$, so its differential dx_p at a point p of S is proportional to $\sum_{j \geq 1} 1_j(p) A_j$. This is a tangent vector to the complex Jacobian variety; in fact, $H[A_j] = O(|\log l_j|^{-1}) = O(j^{-1})$ while $1_j(p) = O(j^{-1})$, by Amplification 6.1, so that $\sum_{j \geq 1} 1_j(p) A_j \in H^\dagger$. The proof that $p \rightarrow x_p$ is an embedding is now finished by noting that dx_p cannot vanish: $dx_p = 0$ requires $1_j(p) = 0$ ($j \geq 1$) and since 1_j ($j \geq 1$) spans H , this would require the vanishing at p of every element of H , which is certainly not the case.

9. The Baker-Its-Matveev formula. The map $M \rightarrow \mathfrak{S}$ is complicated: $q \in M$ determines, first, the points $p_i = (\mu_i, R(\mu_i))$ ($i \geq 1$) of S and then the element ($x = \sum_{i \geq 1} f_{0_i}^{p_i}$ mod periods) of $J^\dagger/L_J = \mathfrak{S}$. Luckily, the inverse map $\mathfrak{S} \rightarrow M$ has a remarkably simple expression in terms of the theta function of §7.

⁴²If $p = (\mu, R(\mu))$, then the point p' complementary to p is $(\mu, -R(\mu))$.

⁴³ $l_j < ae^{-bj}$ when q is real analytic. See Trubowitz [1977: §3] for a proof.

⁴⁴ $K^\dagger + \sqrt{-1} H^\dagger$, modulo periods of that class.

Its-Matveev [1975] discovered this for finite genus; a deeper formula for general hyperelliptic surfaces of finite genus can be found in Baker [1897: 323]. The present extension to infinite genus is new. Let $v_1 = 2c_0 \in J^\dagger$ be the direction corresponding to the infinitesimal translation \mathbf{V}_1 . ϖ is positive on \mathfrak{S} , by Theorem 7.3; it is also infinitely differentiable in the v_1 -direction, by Theorem 7.2.

THEOREM 1 (BAKER'S FORMULA FOR HILL'S SURFACES). *Let $x = \sum_{i \geq 1} \int_{v_i}^{v_i}$ be the general point of K^\dagger , the points v_1, v_2, \dots being in real position on S . Then*

$$0 = \frac{\varpi'(x)}{\varpi(x)} + x(\Delta'/2) \equiv \frac{\varpi'(x)}{\varpi(x)} + \sum_{i \geq 1} \int_{v_i}^{v_i} \frac{\Delta'(\lambda)}{2R(\lambda)} d\lambda,$$

the prime designating differentiation in the v_1 -direction.

The Its-Matveev formula adapted to infinite genus is

THEOREM 2. $q(\xi) = -2(d^2/d\xi^2)\log \varpi(x + \xi v_1)$ ($0 < \xi < 1$), $x \in \mathfrak{S}$ being the point corresponding to $q \in M$.

COROLLARY 1. *Let $v_2 = 4c_1 + 2(\lambda_0 + \sum_{n \geq 1} l'_n)c_0$, as in Theorem 4.4. Then*

$$q(\xi, t) = -2(d^2/d\xi^2)\log \varpi(x + \xi v_1 + t v_2)$$

is the solution of the Korteweg-deVries equation

$$\partial q / \partial t = 3q \partial q / \partial \xi - \left(\frac{1}{2}\right) \partial^3 q / \partial \xi^3.$$

PROOF OF THE ITS-MATVEEV FORMULA. It suffices to check the formula for $\xi = 0$; the rest follows by translation. Let $y_2(1, \lambda) = \prod_{i \geq 1} (\mu_i - \lambda)(i\pi)^{-2}$. Then

$$\mu'_i = \mathbf{V}_1 \mu_i = \frac{2R(\mu_i)}{y_2(1, \mu_i)} \quad (i \geq 1),$$

so by Baker's formula,

$$-2[\log \varpi(x)]'' = \sum_{i \geq 1} \frac{\Delta'(\mu_i)}{R(\mu_i)} \mu'_i = 2 \sum_{i \geq 1} \frac{\Delta'(\mu_i)}{y_2(1, \mu_i)}.$$

Now $\Delta'(\lambda)$ does not belong to J , but $\Delta'(\lambda) + y_2(1, \lambda)$ does since

$$y_2(1, \lambda) = \sin \sqrt{\lambda} / \sqrt{\lambda} + O(\lambda^{-3/2}),$$

while $\Delta'(\lambda) = -\sin \sqrt{\lambda} / \sqrt{\lambda} + O(\lambda^{-3/2})$, and the roots λ_i ($i \geq 1$) of $\Delta'(\lambda) = 0$ interlace the periodic spectrum: $\lambda_{2i-1} < \lambda_i < \lambda_{2i}$ ($i \geq 1$). Thus,

$$\begin{aligned} \sum_{i \geq 1} \frac{\Delta'(\mu_i)}{y_2(1, \mu_i)} &= \lim_{\lambda \downarrow -\infty} \lambda \sum_{i \geq 1} \frac{\Delta'(\mu_i) + y_2(1, \mu_i)}{y_2(1, \mu_i)} \frac{1}{\lambda - \mu_i} \\ &= \lim_{\lambda \downarrow -\infty} \lambda \left[\frac{\Delta'(\lambda)}{y_2(1, \lambda)} + 1 \right] \\ &= \lim_{\lambda \downarrow -\infty} \lambda \left[1 - \prod_{i \geq 1} \frac{\lambda - \lambda_i}{\lambda - \mu_i} \right] \\ &= \sum_{i \geq 1} (\lambda_i - \mu_i). \end{aligned}$$

The proof is finished by use of the trace formula ⁴⁵ $\lambda_0 + \sum_{i>1}(\lambda_{2i-1} + \lambda_{2i} - 2\mu_i) = q(0)$. The additive constant $\lambda_0 + \sum_{i>1}(\lambda_{2i-1} + \lambda_{2i} - 2\lambda_i)$ appears in the result; this is seen to equal $H_0 = \int_0^1 q = 0$ by translating and averaging over a period $0 < \xi < 1$.

PROOF OF BAKER'S FORMULA. Let p_2, p_3, \dots be fixed in real position on S , let $x = \sum_{i>1} \int_{o_i}^{p_i}$, and let

$$f(p_1) = \frac{\varphi'(x)}{\varphi(x)} + \frac{1}{2} x(\Delta') = \frac{\varphi'(x)}{\varphi(x)} + \frac{1}{2} \sum_{i>1} \int_{o_i}^{p_i} \frac{\Delta'(\lambda)}{R(\lambda)} d\lambda$$

for general $p_1 \in S$. Baker's formula states that $f(p_1)$ vanishes identically; in particular,

$$\frac{\varphi'(x)}{\varphi(x)} = - \int_{o_1}^{p_1} \frac{\Delta'(\lambda)}{2R(\lambda)} d\lambda + O(1) = -\sqrt{-\lambda} + O(1)$$

on coverings of nice circles. We now show that Baker's formula would follow from this estimate.

Step 1. $f(p_1)$ is locally analytic on S .

PROOF. $\varphi(x)$ vanishes simply at p'_2, p'_3, \dots and no place else on S , by Theorem 8.2. Let $\lambda(p)$ be the projection of p to the plane and suppose, for the moment, that p_2 is not a branch point. Then, in the vicinity of $p_1 = p'_2$, $\varphi(x) = (\lambda(p_1) - \lambda(p'_2))c$ with $c = c(p_1, p_2, \dots) \neq 0$, and $\varphi'(x)$ vanishes at $p_1 = p'_2$ since $\mathbf{V}_1\lambda(p_1) = \mathbf{V}_1\lambda(p'_2)$ at that point. Clearly, $\varphi'(x)$ also vanishes if p_2 is a branch point, so $\varphi'(x)/\varphi(x)$ is analytic at $p_1 = p'_2$ and, likewise, at p'_3 , etc.

Step 2. $f(p_1)$ is single-valued on S .

PROOF. Let p_1 make a clockwise circuit once about the segment $[\lambda_{2i-1}, \lambda_{2i}]$. By Theorem 7.1, $f(p_1)$ is augmented by $A_i(\Delta'/2) = 2 \int_{\lambda_{2i-1}}^{\lambda_{2i}} R^{-1} d\Delta/2 = 0$. Thus, $f(p_1)$ is real periodic; it is required to prove that it is imaginary periodic, too. Let p_1 make a counterclockwise circuit once about the segment $[\lambda_0, \lambda_{2i-1}]$. By Theorem 7.1, $f(p_1)$ is augmented by $-2\pi\sqrt{-1} v_1(1_i) + B_i(\Delta'/2)$. This is shown to vanish by explicit calculation in the next two lemmas.

LEMMA 1. $B_i(\Delta'/2) = 2\pi\sqrt{-1} \times i (i > 1)$.

PROOF. The actual determination of the radical $R(\lambda)$ is important now: it is taken real and positive for $\lambda < \lambda_0$ and determined elsewhere by analytic continuation off the cut $[\lambda_0, \infty)$, as in Figure 5.

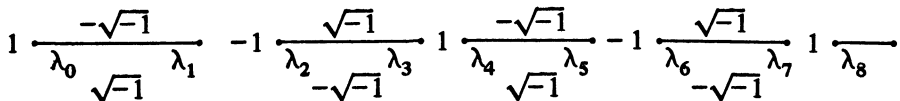


FIGURE 5

Thus,

⁴⁵See McKean-van Moerbeke [1975: 254-258].

$$\begin{aligned}
 B_i(\Delta'/2) &= \oint \frac{\Delta' d\lambda}{2R} \\
 &= -2\sqrt{-1} \int_{\lambda_0}^{\lambda_1} \frac{\Delta'(\lambda)}{\sqrt{4 - \Delta^2(\lambda)}} d\lambda + 2\sqrt{-1} \int_{\lambda_2}^{\lambda_3} \frac{\Delta'(\lambda)}{\sqrt{4 - \Delta^2(\lambda)}} d\lambda - \dots \\
 &= -2\sqrt{-1} \int_2^{-2} \frac{d\Delta}{\sqrt{4 - \Delta^2}} + 2\sqrt{-1} \int_{-2}^2 \frac{d\Delta}{\sqrt{4 - \Delta^2}} - \dots \\
 &= -2\pi\sqrt{-1} \times i.
 \end{aligned}$$

LEMMA 2. $v_1(1_i) = i$ ($i \geq 1$).

PROOF. Under translation through a period $0 \leq \xi < 1$, p_i makes i full circuits about the segment $[\lambda_{2i-1}, \lambda_{2i}]$. Thus, for $\phi \in J$,

$$v_1(\phi) = \sum_{j>1} 2j \int_{\lambda_{2j-1}}^{\lambda_{2j}} d\Phi = \sum_{j>1} jA_j(\phi).$$

The evaluation follows at once.

Step 3. $f(p_1)$ has no pole at $p_1 = \infty$; in fact, $f(p_1)$ vanishes identically.

PROOF. The estimate of ω'/ω now comes into play: $f(p_1)$ is seen thereby to be bounded on the covering of nice circles. Now $f(p_1)$ is necessarily of the form $a(\lambda) + R(\lambda)b(\lambda)$ with $\lambda = \lambda(p_1)$ and integral a, b ; in particular, $2a(\lambda) = f(p_1) + f(p'_1)$ is bounded and so constant, as is $4R^2(\lambda)b^2(\lambda) = [f(p_1) - f(p'_1)]^2$. Thus, $f(p_1)$ is constant. Now if p_1 , like p_2, p_3, \dots , is placed in real position, then $\omega'(x)/\omega(x) + x(\Delta'/2)$ is symmetrical in p_i ($i \geq 1$); as such, it cannot depend on any of these variables, and its value, namely 0, is deduced by taking $p_i = o_i$ ($i \geq 1$). Then $x = 0$, $\omega(x)$ is at its maximum value on \mathfrak{S} , and since v_1 is tangent to \mathfrak{S} , $\omega'(0) = 0$. The proof is finished.

Unfortunately, the estimate of ω'/ω in the vicinity of $p_1 = \infty$ is not directly available, and it is necessary to pass to the limit from finite to infinite genus in the style of the proof of Theorem 8.1. To do this, let $S_g, R_g^2(\lambda) = -\prod_{i=1}^g (\lambda - \lambda_i), H_g, \Theta_g, \mathfrak{S}_g$, etc. be as before, excepting the temporary omission of the convergence factor $\pi^{-4g}(g!)^{-4}$ from the second item. Let v_g^ξ be the direction tangent to \mathfrak{S}_g defined by $v_g^\xi(\phi) =$ the coefficient of λ^{g-1} for polynomials ϕ of degree $< g$, let p_2, \dots, p_g be fixed in, e.g., real position on S , and for p_1 in general position, let $x = \sum_{i=1}^g \int_{o_i}^{p_i}$. Let A_i ($i = 1, \dots, g$) be the real periods for S_g , let the normalized basis $d\Phi_j$ ($j = 1, \dots, g$) of differentials of the first kind be as in §8, and let Δ_g be the unique polynomial $\lambda^g + c_1\lambda^{g-1} + \dots$ such that $A_i(\Delta_g) = 0$ ($i = 1, \dots, g$).⁴⁶ Baker's formula for finite genus asserts the vanishing of

$$f(p_1) = \frac{\Theta'_g(x)}{\Theta_g(x)} + \sum_{i=1}^g \int_{o_i}^{p_i} \frac{\Delta_g d\lambda}{2R_g},$$

the prime designating differentiation in the v_g^ξ -direction. Let $d\Psi_g$ be the normalized differential of the second kind $(\Delta_g/2)R_g^{-1} d\lambda$.

⁴⁶ $\Delta_g(\lambda) = \lambda^g - \sum_{i=1}^g A_i(\lambda^g) 1_i^\xi(\lambda)$.

Step 1 is performed as for infinite genus.

Step 2 is also the same, except for the proof of the imaginary periodicity. This requires

LEMMA 3.⁴⁷ $-2\pi\sqrt{-1} v_1^g(1^g) + B_i(\Delta_g/2) = 0$ ($i = 1, \dots, g$).

PROOF. If the statement of the lemma is correct, then for any polynomial ϕ of degree $< g$

$$2\pi\sqrt{-1} v_1^g(\phi) = 2\pi\sqrt{-1} \sum_{i=1}^g A_i(\phi) v_1^g(1^g) = \sum_{i=1}^g A_i(\phi) B_i(\Delta_g/2).$$

To prove this equivalent statement, let B , C , and D be as in Figure 3 with $n = g$ and $\lambda_{2g} < r < \infty$. Then, for any differential $d\Phi$ of the first kind,

$$\begin{aligned} 0 &= \int_D d\Psi_g \wedge d\Phi = \int_B \Psi_g d\Phi \\ &= \int_C \Psi_g d\Phi + \sum_{i=0}^{g-1} \int_{\lambda_{2i}}^{\lambda_{2i-1}} [\Psi_g(+)+\Psi_g(-)] d\Phi \\ &\quad + \int_{\lambda_{2g}}^r [\Psi_g(+)+\Psi_g(-)] d\Phi \\ &\quad + \sum_{i=1}^g \int_{\lambda_{2i-1}}^{\lambda_{2i}} [\Psi_g(+)-\Psi_g(-)] d\Phi, \end{aligned}$$

the integrals in the sum being taken on the upper bank of the cut. The proof is finished by evaluating these contributions, separately. The first integral is easy: $R_g \sim \varepsilon \lambda^{g+1/2}$ on C with a root of unity $\varepsilon = \pm\sqrt{-1}$, so

$$\Psi_g(\lambda) \sim \varepsilon^{-1} \int_0^\lambda \frac{d\lambda}{\sqrt{\lambda}} = \frac{2}{\varepsilon} \sqrt{\lambda}, \quad d\Phi(\lambda) \sim v_1^g(\phi) \varepsilon^{-1} \lambda^{-3/2} d\lambda,$$

and the integral contributes

$$o(1) + \varepsilon^{-2} v_1^g(\phi) \int_C \frac{d\lambda}{\lambda} = o(1) - \pi\sqrt{-1} v_1^g(\phi).$$

Now inspection of Figure 3 shows that the integrands $\Psi_g(+)+\Psi_g(-)$ in the first sum vanish by reason of $A_i(\Delta_g) = 0$ ($i = 1, \dots, g$), so this sum contributes nothing. The second sum is now evaluated as $(1/2)\sum_{i=1}^g A_i(\phi) B_i(\Delta_g/2)$ by noting that $\Psi_g(+)-\Psi_g(-) = B_i(\Delta_g/2)$ in $[\lambda_{2i-1}, \lambda_{2i}]$ ($i = 1, \dots, g$). The stated identity now appears by making $r \uparrow \infty$.

Step 3 is an elementary estimate. $\Theta_g(x)$ vanishes simply at $p_1 = \infty$ by the vanishing theorem of Baker [1897: 309]. Thus, $f(p_1)$ has at worst a pole at ∞ , and this can be detected by letting $\lambda(p_1) = \mu_1 \downarrow - \infty$: The simple vanishing of $\Theta_g(x)$ at $p_1 = \infty$ implies $\Theta_g(x) \sim c\mu_1^{-1/2}$ and $\Theta'_g(x)/\Theta_g(x) \sim -(1/2)\mu'_1/\mu_1$ in the vicinity; it is required to estimate this quantity. To do this, compute μ'_1 by differentiating $x(\phi) = \sum_{i=1}^g \int_{v_i^0} v_i^g d\Phi$ in the v_i^g -direction and substituting $\phi_1(\lambda)$

⁴⁷ B_j ($j = 1, \dots, g$) are the imaginary periods for S_g .

= $\prod_{j \neq 1} (\lambda - \mu_j)(\mu_1 - \mu_j)^{-1}$. This produces $\mu'_1 = R_g v_1^g(\phi_1) \sim (-1)^{g-1} (-\mu_1)^{3/2}$. Thus, $\Theta'_g/\Theta_g \sim (-1)^g (-\mu_1)^{-1/2}$ as $\mu_1 \downarrow -\infty$. This is to be compared to

$$\int_{o_1}^{p_1} d\Psi_g \sim (-1)^g \int_0^{\mu_1} \frac{d\lambda}{\sqrt{-\lambda}} = (-1)^{g-1} (-\mu_1)^{-1/2},$$

the upshot being that $f(p_1)$ has no pole at ∞ . The rest of Step 3 is as before. The proof of Baker's formula for finite genus is finished.

To carry Baker's formula over to infinite genus, it is necessary to reintroduce the normalizing factor $\pi^{4g} (g!)^{-4}$ into R_g^2 and to compensate by the factor $\pi^{-2g} (g!)^{-2}$ in Δ'_g .

LEMMA 4. *Let Δ'_g be so renormalized. Then it is possible to make $g \uparrow \infty$ in such a way that Δ'_g converges to Δ' locally uniformly.*

PROOF. $A_i(\Delta'_g) = 0$ ($i = 1, \dots, g$) implies that $\Delta'_g(\lambda) = 0$ has a root λ_i in each of the intervals $[\lambda_{2i-1}, \lambda_{2i}]$ ($i = 1, \dots, g$) and no others. Thus, $\Delta'_g(\lambda) = \prod_{i=1}^g (i^2 \pi^2)^{-1} (\lambda - \lambda_i)$ converges, locally uniformly, to a function $\psi(\lambda)$ of the general form $-y_2(1, \lambda)$ with⁴⁸ $A_i(\psi) = 0$ ($i = 1, 2, \dots$). The identification of ψ with Δ' is now accomplished by noting that $\psi - \Delta'$ is of class $I^{3/2}$ and so vanishes by reason of $A_i(\psi - \Delta') = 0$ ($i = 1, 2, \dots$) and Corollary 5.1.

LEMMA 5.

$$\lim_{g \uparrow \infty} \sum_{i=1}^g \int_{o_i}^{p_i} d\Psi_g = \sum_{i>1} \int_{o_i}^{p_i} \Delta'(\lambda) R^{-1} d\lambda$$

for $g \uparrow \infty$ in the manner of Lemma 4.

PROOF. The details are left to the reader. It is necessary to estimate the tail of the left-hand sum; this is facilitated by assuming that the path from o_i to p_i winds not more than once about the circle covering $[\lambda_{2i-1}, \lambda_{2i}]$.

Now, by §8, $\Theta_g(x)$ and $x_g = \sum_{i=1}^g \int_{o_i}^{p_i}$, tend to their analogues for infinite genus as $g \uparrow \infty$ in any manner. The proof of Baker's formula is finished by confirming that⁴⁹

$$\Theta'_g(x) = \sum 2\pi\sqrt{-1} v_1^g(\phi) e^{2\pi\sqrt{-1} x_g(\phi) - (\pi/2)H_g[\phi]}$$

does the same as $g \uparrow \infty$ in the manner of Lemma 4. Now the renormalization introduced a factor $\pi^{2g} (g!)^2$ into v_1^g , so, by Lemma 3, $2\pi\sqrt{-1} v_1^g(1_j^g) = B_j(\Delta'_g/2)$ with the renormalized Δ'_g . Thus, by Lemmas 1, 2, and 4, $2\pi\sqrt{-1} v_1^g(1_j^g)$ tends to its analogue $2\pi\sqrt{-1} v_1(1_j) = B_j(\Delta'/2)$, for infinite genus. The final step is to control the tail of the sum. The rest of the proof will be plain from §8. Let μ_i ($i = 1, \dots, g$) be as in Lemma 8.1. $v_1^g(1_j^g)$ is the same, normalized or not; for the present, the unnormalized form is best, leading easily to the appraisal

$$|v_1^g(1_j^g)| = O(j) \times \prod_{\substack{1 < i < g \\ i \neq j}} \frac{\sqrt{(\lambda'_j - \lambda_{2j-1})(\lambda'_j - \lambda_{2i})}}{|\mu_j - \mu_i|} = O(j)$$

⁴⁸ A_i ($i > 1$) are now the real periods for infinite genus.

⁴⁹The sum is taken over $\phi = \sum_{i=1}^g n_i 1_i^g$, $n \in Z^g$.

with $O(j)$ independent of g and λ_j' intermediate between λ_{2j-1} and λ_{2j} . Now let p_1 , like p_2, p_3, \dots , be in real position and let $|n| = \sum_{i=1}^g i|n_i|$. Then

$$\sum_{|n| > m} |v_1^g(\phi) e^{2\pi\sqrt{-1} x_g(\phi) - (\pi/2)H_g[\phi]}|$$

is bounded, as in §7, by a multiple of

$$\begin{aligned} \frac{1}{m} \sum_{\substack{|n| > m \\ \text{tame } n}} |n|^2 \prod_{i > 1} l_i^{n_i^2} &< \frac{1}{m} \sum_{\text{tame } n} \prod_{i > 1} (1 + i|n_i|)^2 l_i^{n_i^2} \\ &< \frac{1}{m} \prod_{i > 1} \sum_{n=-\infty}^{\infty} (1 + i|n|)^2 l_i^{n^2} \\ &= O(1/m), \end{aligned}$$

independently of g . The proof of Baker's formula for infinite genus is finished.

10. Meromorphic functions. A meromorphic function on S is of the form $f(p) = a(\lambda) + R(\lambda)b(\lambda)$, in which a and b are meromorphic on the plane. Naturally, without some control on $f(p)$ in the vicinity of $p = \infty$, nothing more can be said about such functions. The adjective "meromorphic" is therefore construed from now on in a narrower sense: in terms of the *local parameter* $\zeta = \lambda^{-1/2}$, it is required that $f(p) \sim c_n \zeta^n + \dots$ on the double coverings of nice circles. f is said to have a root (pole) at $p = \infty$ of multiplicity $m = |n|$ if $n \geq 0$ ($n < 0$) and $c_n \neq 0$.

EASY HALF OF ABEL'S THEOREM. Let f be meromorphic on S , let $d\Phi$ be of class $I^{3/2}$ or better, and let⁵⁰ $\Phi(p) = \int_0^p d\Phi$. Then, modulo periods,

$$\sum_{f(q)=0} \Phi(q) - \sum_{f(p)=\infty} \Phi(p) + n\Phi(\infty) = 0.$$

AMPLIFICATION 1. The sum requires clarification. $\Phi(p)$ is single-valued only on the universal covering of S , so periods intervene. The correct statement is that, if $x_p = \int_0^p$, then $\sum x_q - \sum x_p + nx_\infty = 0$, the possibly infinite sum being taken in \mathfrak{S} , i.e., modulo periods of class $I^{3/2}$, summing first over the roots and poles inside the covering of a nice circle and then letting the radius of this circle approach infinity.

AMPLIFICATION 2. Presumably, $\sum \Phi(q) - \sum \Phi(p) + n\Phi(\infty) = 0$, properly interpreted modulo periods, is also a sufficient condition for the existence of a meromorphic function with divisor $\sum(q - p) + n\infty$, but this is not proven here.

PROOF. Let B_1 be a nice circle C_1 of radius r with a 2-banked cut along $[0, r]$, and small detours to avoid the roots and poles of f , and let B_2 be its double covering on S . Then $\Phi(p)$ is single-valued inside B_2 , and

$$\frac{1}{2\pi\sqrt{-1}} \int_{B_2} \Phi(p) d \log f(p) = \sum \Phi(q) - \sum \Phi(p) = s(r)$$

the sum being taken over the roots q and poles p of f inside B_2 . The integral is

⁵⁰ $0 = (\lambda_0, 0)$.

now divided into a part coming from the covering C_2 of C_1 and a part coming from the covering of the cut. Now $\phi \in I^{3/2}$, so $|\phi(\lambda)| |R(\lambda)|^{-1} = O(r^{-5/4})$ on C_1 , by an estimate similar to that employed in the proof of Theorem 4.2. Thus, on C_2 , $|\Phi(p) - \Phi(\infty)| = O(r^{-1/4})$, while $d \log f(p) = O(\lambda^{-1} d\lambda) = O(d\theta)$, so that

$$\int_{C_2} |\Phi(p) - \Phi(\infty)| |d \log f(p)| = O(r^{-1/4})$$

can be neglected, with the result that

$$\frac{1}{2\pi\sqrt{-1}} \int_{C_2} \Phi(p) d \log f(p) = O(r^{-1/4}) - n\Phi(\infty).$$

The proof is finished by evaluating the integral along the covering of the cut. This contributes

$$\frac{1}{2\pi\sqrt{-1}} \int [\Phi(+)-\Phi(-)] d \log f(p);$$

the integral is extended along the double covering of the upper bank, and $\Phi(+)$ [$\Phi(-)$] is the value of $\Phi(p)$ on the upper [lower] bank. Now $\Phi(+)-\Phi(-)$ is a period of $d\Phi$, alternately vanishing and constant between the points λ_i ($i > 0$). Thus, the cut contributes a sum of the form

$$\sum_{i=1}^n \text{period} \times \frac{1}{2\pi\sqrt{-1}} \int d \log f(p),$$

the i th integral being taken about a double covering of the segment $[\lambda_{2i-1}, \lambda_{2i}]$. This circuit returns $f(p)$ to its initial value, so each integral contributes an integer, i.e., the cut contributes a period, and the upshot is that, modulo periods of class $I^{3/2\uparrow}$, $s(r) + n\Phi(\infty) = O(r^{-1/4})$. The proof is finished.

DEGREE. The same procedure with 1 in place of $\Phi(p)$ leads to the existence of the degree of a meromorphic function: with a technical interpretation, as in Amplification 1, *a meromorphic function has the same number of roots as poles*; in particular, a pole-free function is constant.

RIEMANN-ROCH THEOREM. The classical theorem of Riemann-Roch states that if $\delta = p_1 + \dots + p_d$ is any divisor on a closed Riemann surface of finite genus g , if F is the class of meromorphic functions having the same or softer poles, and if D is the class of complex differentials of the first kind having the same or harder roots, then⁵¹ $\dim F = d + 1 - g + \dim D$. A number of simple examples will suggest that this relation may be interpreted for meromorphic functions construed as above and differentials of the first kind from the complexification of K .

EXAMPLE 1. Let p' be the point complementary to $p \in S$ and let $\lambda(p)$ be its projection to the plane. The divisor $\delta = p + p'$ is of degree 2, while $\dim F = 2$: 1 dimension for the constants and 1 for the function $f(p) = (\lambda - \lambda(p))^{-1}$ with simple poles at p and p' . The Riemann-Roch relation predicts $g - \dim D$

⁵¹ \dim is the complex dimension.

= 1. Now it is natural to interpret the left side as the (complex) codimension of D in the complexification of K . This is, in fact, correct: $d\Phi \in D$ if and only if ϕ vanishes at the common projection λ of p and p' . This cuts the total dimension ' $g = \infty$ ' down by 1, the evaluation map $\phi \rightarrow \phi(\lambda)$ being an element of the complexification of K^\dagger .

EXAMPLE 2. Let p_i ($i \geq 1$) be in real position on S and let $\delta = p_2 + p'_2 + p_3 + p'_3 + \dots$. Then δ is of degree ' $2g - 2$ ', $\dim D = 1$,⁵² and the Riemann-Roch relation predicts $\dim F = '2g - 2 + 1 - g + 1' = 'g'$. This is confirmed by noting that the ' $g - 1$ ' functions $f_i(p) = (\lambda - \lambda(p_i))^{-1}$ ($i \geq 2$) are independent and, together with the constants, presumably account for the whole of F .

EXAMPLE 3. Let p_i ($i \geq 1$) be in real position on S , as before. The divisor $\delta = p_1 + p_2 + \dots$ is of degree ' g '. Now the roots of a differential of the first kind come in complementary pairs, so $\dim D = 0$, by interpolation. The Riemann-Roch relation predicts $\dim F = 'g + 1 - g + 0' = 1$, and that is correct: the poles of a meromorphic function come in complementary pairs, so functions of class F are pole-free, including at the point ∞ . But such a function is constant, i.e., $\dim F = 1$.

These examples suggest that the Riemann-Roch theorem still has much to say. Doubtless, it is always correct, properly construed, and it is natural to hope that this construction will make itself apparent upon reconsidering the proof, giving the *geometrical* facts underlying the dimension counts precedence over the mere *numerical* dimensions.

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⁵² $d\Phi(p) = \prod_{j \neq 1} (\lambda - \lambda(p_j))(\lambda(p_1) - \lambda(p_j))^{-1} R(\lambda)^{-1} d\lambda$ is of class D .

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