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BOOK REVIEWS

On numbers and games, by J. H. Conway, Academic Press, London, New York, San Francisco, 1976, ix + 238 pp., \$26.50.

Surreal numbers, by D. E. Knuth, Addison-Wesley, Reading, Massachusetts, 1974, 119 pp.,

And more than everything my son, o beware:
the making of many books without end;
and excessive studiousness, tiredness of flesh.

(Translated by Bill from Ecclesiastes XII, 12.)

Some readers know to play the game of nim well, fewer play a perfect annihilation game, and nobody *knows* whether there exists an opening move in chess that will *guarantee* a win for white. These games and many more, belong to the family of *combinatorial games*, by which we mean the set of all two-player perfect-information games without chance moves and with outcomes lose or win (and sometimes: dynamic tie). The motivation for ONAG may have been, and perhaps was–and I would like to think that it was–the attempt to bridge the theory gap between nim-like and chess-like games.

Why is there a gap?

Every combinatorial game can be described as a directed graph called *game-graph*, whose vertices are the game positions, and (u, v) is a directed edge if and only if there is a move from position u to position v. Denote by N the set of all positions from which the Next (first) player can force a win; by P the set of all positions from which the Previous (second) player can force a win; and by T the set of all (dynamic) Tie positions, which are positions from which no player can force a win and therefore both can avoid losing. In an acyclic game-graph there cannot be any tie positions. The N, P, T classification of any game graph $R = (V, E)$ can be determined in $O(|V| + |E|)$ steps [**8**]. For both nim and chess, a finite game-graph can be constructed and the N, P, T classification can be determined. So both games are solvable *in principle*.

If we play nim with n piles, each pile containing at most k tokens, then the game-graph contains $(k + 1)^n$ vertices. Suppose that in (generalized) chess played on an $n \times n$ board there are k different pieces. If k is about $n^2/2$, then the game-graph of chess contains $O(2^{n^2})$ vertices. So both game-graphs have exponentially many vertices, and thus both games *appear* intractable in the usual sense of computational complexity [**1**, Chapter 10], [**14**, Chapter 9], namely a computation appears to be required which is asymptotically exponential.

From a computational efficiency standpoint, the essential difference between nim and chess is that nim can be viewed as a *disjunctive compound* (sum) of independent games, namely the individual piles. A disjunctive compound of games is a finite collection of games. Each player at his turn selects one game and makes a legal move in it. The player first unable to make a move is the loser. Applied to nim, this enables replacement of the game-graph containing $(k + 1)^n$ vertices by the game-graph of a single pile, which makes the strategy of nim tractable, i.e., polynomial in n. But for chess-like games no reduction of game-graphs is known. Therefore they appear intractable at present. In fact, some combinatorial games **[9]**, **[16]** can be proved to be *NP*-hard (terminology defined in **[14**, Chapter 9**]**). But this *complexity approach* is not at all pursued in ONAG, in which inductive constructions are the order of the day.

What enables the replacement of large by small game-graphs in nim-like games, such as those considered in **[12]**, is a tool called the (classical) Sprague-Grundy function. The tool is useful for combinatorial games without dynamic ties in which the first player unable to move is the loser, the other the winner. This tool breaks down for the more complex games. Here are some of the properties of the more sophisticated games which damage or destroy the existence or applicability of the (classical) Sprague-Grundy function.

I. *Existence of dynamic ties*. A polynomial strategy can be recovered for this case by defining a *generalized* Sprague-Grundy function **[6]**, **[8]**, **[15]**. This theory is illustrated in two and a half pages in ONAG by means of a sample game called "Traffic Jams".

II. *Various interactions between game tokens, including jump and capture rules*. An example is *Welter's game* (Chapter 13 of ONAG), which is played on a semi-infinite linear board with a finite number k of coins, at most one per square (Figure 1). The squares are numbered 0, 1, 2, ... from the left. Either player at his turn selects a coin and moves it to *any* unoccupied square with a lower number. In particular, any coin is allowed to bypass other coins. The game ends when the loser is unable to move because the coins are jammed in positions $0, 1, \ldots, k - 1$.

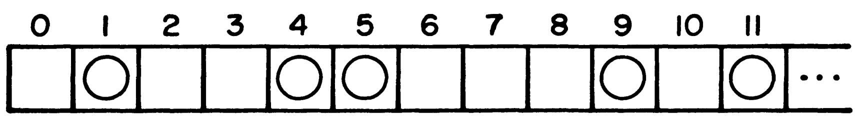

FIGURE 1. A typical position in Welter's game

Welter gave a partial analysis of the game **[17]**. Conway gives a complete theory (except for one open problem of secondary importance) which is rather curious, based on a *mating* function and an *animating* function (named after *a*ddition, *nim*-addition and preserving the *mating* function), which uses both ordinary and nim-addition.

Another example where tokens interact is the set of *annihilation games*, in which two colliding tokens get annihilated and go out of the game. Their game-graphs can be replaced by sufficiently small graphs (in the form of so-called *contrajunctive compounds* rather than disjunctive compounds), so that the strategy becomes polynomial if somewhat intricate **[7]**.

III. *Partizanship of games.* A game is called *impartial* if the options of both players are identical for each position in the game. Otherwise it is *partizan.* Nim-like games are impartial; checkers, chess and go are partizan. It is the area of partizan games in which the main impact of ONAG lies, and from which the motivation for SUN emerges.

Normally in a review, one can refer to the book's theory x and Theorem y, with which the readership is reasonably familiar, and then comment about the author's approach in selecting them, illuminating them, fusing them together, proving them, etc. ONAG, on the other hand, is a highly original research book (including many applications), and its main ideas have not yet been disseminated widely. Therefore we give below a very brief and informal summary of Conway's theory of partizan games, which will sometimes be called simply *games* below.

Games are defined inductively. If G^L and G^R are any two sets of games, then there is a game $\{G^L|G^R\}$. All games are constructed in this way.

Numbers constitute a subclass, and they are also defined inductively: if N^L and N^R are any two sets of numbers, and no member of N^L is $\geqslant$ any member of N^R, then there is a number $\{N^L|N^R\}$. All numbers are constructed in this way.

The empty set $\{\ \}$ serves to define the number created on day 0: $\{\ |\ \} = 0$. On day 1, the numbers $\{\ |0\} = -1$, $\{0|\ \} = 1$ and the game $\{0|0\} = *$ are created, and on day 2 we get, among other numbers, $\{0|1\} = \frac{1}{2}$, and among other games, $\{1|-1\} = \pm 1$, $\{0|*\} = \uparrow$ (see the "partizan game-graphs" in Figure 2, in which vertices are game positions, and edges slanted in south-westerly direction denote moves of Left; in south-easterly direction moves of Right). Note that these special cases obey the following rules, which turn out to hold in general: $G > 0$ if Left can win (i.e., can make the last move); $G < 0$ if Right can win; $G = 0$ if the second player can win; $G|\,|0$ (G is *fuzzy*) if the first player can win. Because of these properties, the determination of the value of G of a partizan game is of fundamental importance to the theory.

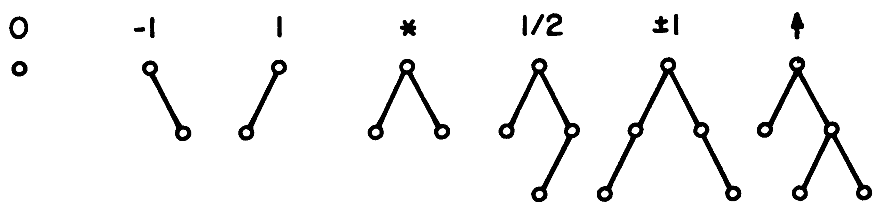

FIGURE 2. A few partizan games

The fuzzy games are those which are neither positive nor negative nor zero. For example $\{x|y\}$ with $x \not\geqslant 0 \not\geqslant y$ is fuzzy. There are also *hot* and *cold* games. For example $\{x|y\}$ with $x > y$ is hot (with *temperature* $x - y$), and all numbers are cold. The temperature theory, complete with thermographs and cooling devices, is designed to bound the values of hot games. The important Simplicity Theorem (rather: a special case thereof) determines the values of numerical games. It states that if N^L and N^R are sets of numbers and $\{N^L|N^R\}$ is a number, then it is the *simplest* number N which is $>$ all numbers in N^L and $<$ all numbers in N^R. The order of simplicity is: integers with increasing absolute value, followed by the dyadic rationals with increasing denominators, followed by all other numbers.

In a disjunctive compound of two games G, H, a player selects one game and makes a legal move in it. This suggests the following inductive definition for the sum:

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}.$$

If we define $-G = \{-G^R | -G^L\}$, we can verify (e.g., Figure 2) that $\frac{1}{2} + \frac{1}{2} = 1$, in the sense that $\frac{1}{2} + \frac{1}{2} - 1 = 0$, i.e., the second player can win the disjunctive compound of the games $\frac{1}{2}$, $\frac{1}{2}$, -1. (Now show that $\{0|\uparrow\} = \uparrow + \uparrow + *$.) Also multiplication can be defined.

Games are ordered by their birthday and partially ordered by $<$. The numbers created on the finite days comprise the set of all dyadic rationals. On day ω we get, for example, the ordinal number $\omega = \{0, 1, 2, \ldots | \}$, the infinitesimal number $1/\omega = \{0 | 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, the transcendental numbers π, e, and $\{0.1, 0.101, 0.10101, \ldots | 0.11, 0.1011, 0.101011, \ldots\} = x$ (in binary notation), which turns out to be $\frac{1}{3}$. (Verify that $x + x + x = 1$.)

The game $\uparrow$ is positive (Left can win!) but it is smaller than every positive infinitesimal *number*. In particular, $\uparrow < 1/\omega$. One gets an exciting and weird world of games. An example is the game *domineering*, played with dominoes each of which covers precisely two squares of an $n \times n$ chessboard. Left places his dominoes vertically, Right horizontally. Dominoes must not overlap, and the player first unable to move is the loser. Note that for example for $n = 2$, the game has the hot value ± 1. Either player, by playing in ± 1, reserves a further move for himself, while his opponent is unable to move.

After a while of playing domineering, the board may split into several disconnected regions, and the game becomes a disjunctive compound of several games. It is best for each player to move in a hottest game.

Left-Right Hackenbush, also described in **[10]**, is a cold game played on a graph whose vertices are painted Lilac and Red. At least one vertex in each component is grounded. Left (Right) deletes a Lilac (Red) edge together with all vertices no longer connected to ground, and their incident edges (of either color). The game has a very curious polynomial strategy if the graph is a tree. But for a general graph, Berlekamp [3] showed that even the special case of bipartite redwood furniture is NP-hard.

If all edges are painted Ivory, the game is impartial. Its theory makes use of a mating-type function. Of course there is also Hackenbush Hotchpotch in which there exist edges of all three colors. For this game wop and sop functions come in handy, which are generalizations of animating functions.

In addition to the analysis of many special families of impartial and partizan games such as the octal games of Guy and Smith of which Kayles is an example **[12]**, tame restive and restless games; even, odd and prime games; shrinking rectangles, twisted Bynum's game and contorted fractions; cutcake, Col, Snort and others–many general theories and algorithms are developed in ONAG. For example, an algorithm is given for determining for any game G whether it is a number or not, and, in the latter case, determining all numbers

x and y satisfying $x \geqslant G$, $y \leqslant G$ by means of two Dedekind sections. Algorithms for simplifying the form of G are also given. The theory of impartial games is deduced as a special case of the theory of partizan games. A theory of misère play (when the last player to move *loses*) is given. In addition to disjunctive compounds, theories for other conventions of playing a set of games are given, such as *conjunctive* compounds (make a move in each component game which has not yet ended) and *selective* compounds (select some of the component games and make a move in each of the selected ones) and many variations including compounded compounds.

In the last few sections a theory of the small world complete with atomic weights, atomic mass thermography and superstars is given, which is designed to bound the values of small games. It was developed jointly by Conway and Simon Norton. A game G is *small* if $-x < G < x$ for every positive *number* x. Since $\uparrow > 0$, $-\uparrow = \downarrow < 0$, $*| \,|0$ and 0 are small games, the small world is a microcosm of the large world. The theory of the small world is not yet complete, and the author concludes with: "We leave these questions to others, who will surely find many other problems to puzzle them and wonders to amaze and amuse them in this curious world of games. Only a certain feeling of incompleteness prompts us to add a final theorem.

THEOREM 100. *This is the last theorem in this book.*

(The proof is obvious.)"

The above encapsulates the flavor of the book, mainly its First (games) Part, which contains the heart of the treatise, but is intimately linked to the Zeroth (numbers) Part. The latter starts with the inductive construction of numbers given above, and the inductive definitions of $\leqslant$, $+$, $-$, $\times$. They generate a real closed field with a very rich structure, though they comprise only the proper subclass of all numbers. These are ordered by their birthday and also by $\leqslant$. The remainder of the Zeroth Part is devoted to proving that it is a field containing the real and ordinal numbers, and to an investigation of various properties of this field, including the Simplicity Theorem.

ONAG is a deep-reaching book. It can be read on at least three levels. It can be read cursorily, just enjoying some of the games and their curious algorithms. It can be read diving into the deeper waters and working out the proofs of the main results. This is hard in some places, since the book is written concisely. And it can be read with the purpose of exploring some of the research problems which suggest themselves almost on every page. The richness of the theory has already induced research activities in a number of fields. Berlekamp has devised a representation of numbers whose integer and fractional parts are expressed in unary and binary respectively, which does not require an explicit binary point and sign bit. This representation arises naturally in describing the value of a Left-Right Hackenbush string, and can be used for the internal number representation in a computer [2]. Li extended the theory of numerical partizan games to include cycles [13]. Berlekamp, Conway and Guy pick up some of the threads of ONAG in *Winning Ways* [3]. Martin Kruskal is exploring and developing the ramifications of Conway's number construction. And Knuth used it in SUN for a unique purpose which we are now going to discuss. However, as Martin Gardner remarked, hard-core set theorists do not appear to have discovered the theory yet.

SUN relates the story of Alice and Bill who ran away from it all to find themselves on the shores of the Indian Ocean. But soon enough they start to crave "for a book to read–*any* book, . . . , even a math textbook." It is therefore not surprising that they stumble across a broken off rock on which the story of the creation is inscribed in Hebrew, which Bill translates as follows: "In the beginning, everything was void, . . . the first rule: Every number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set. . . . And the first number was created from the void left set and the void right set. Conway called this number 'zero', and said that it shall be a sign to separate positive numbers from negative numbers. Conway proved that zero was less than or equal to zero, and he saw that it was good. And the evening and the morning were the day of zero. On the next day, two more numbers were created, one with zero as its left set and one with zero as its right set. And Conway called the former number 'one', and the latter he called 'minus one'. And he proved that minus one is less than but not equal to zero and zero is less than but not equal to one. And the evening . . . " That is where the rock breaks off.

In the next four chapters our two heroes prove that $-1 < 0 < 1$, $x \leqslant x$, the transitive law, the ordering of numbers by $\leqslant$ and other properties. In Chapter 6 they anticipate the events of the third and in fact of the nth day of creation. After they start to wonder in what sense these entities are *numbers*, they find the second part of the rock which Bill again manages to translate from Hebrew. It contains addition and multiplication rules, and: " . . . When the numbers had been created for infinitely many days, the universe itself appeared. And the evening and the morning were $\aleph$ day . . . ". Alice and Bill now continue their inquisitive explorations and prove commutativity and associativity of addition, and properties such as $x + 0 = x$, $x - x = 0$. They decide that the "universe" is the set of numbers created on day ω, like $\frac{1}{3}$, $\omega, \ldots,$ continue to construct numbers created on subsequent days, and discover elementary properties of multiplication.

The message of SUN is not in *what* it presents–it covers less than the first twenty pages of ONAG–but in *how* it presents it. In a postscript Knuth states that the book is intended primarily for college mathematics students at about the sophomore or junior level. He adds, "Of course, I wrote this mostly for fun, . . . but I must admit that I also had a serious purpose in the back of my mind. Namely, I wanted to provide some material which would help to overcome one of the most serious shortcomings in our present educational system, the lack of training for research work. . . . My primary aim is not really to teach Conway's theory, it is to teach how one might go about developing such a theory. Therefore as the two characters in this book gradually explore and build up Conway's number system, I have recorded their false starts and frustrations as well as their good ideas. I wanted to give a reasonably faithful portrayal of the important principles, techniques, joys, passions and philosophy of mathematics, so I wrote the story as I was actually doing the research myself (using no outside sources except a vague memory of a lunchtime conversation I had had with John Conway almost a year earlier)."

This aim is achieved beautifully. The frustrations and joys of Alice and Bill flow forth with a true-life quality. For example, they get entangled in circular arguments while trying to prove the transitive law. Luckily they finally succeed in extricating themselves by discovering induction on the sum of the birthdays of the numbers involved. Or, when after some struggle, they come up with the correct definition of sum, they are struck by disaster when they realize that they neglected to show that the sum is a number! This upsets some of their previous results. While trying to mend this hole, clever Alice is naturally led to the discovery of pseudo-numbers(= games–in Conway's language).

These creative activities lead Bill and Alice to the conclusion that mathematics is such a drag in school because it is presented as a finished product in the Landau telegraphic style of lemma, proof, theorem, proof, ... They advocate, instead, a class atmosphere conducive to the discovery of theorems, or at least an education encouraging one to ask one's own questions and work at one's own solutions to problems presented in a mathematical text-book. Whether such an approach will actually find the wide acceptance it deserves is of course a question which only time will answer.

SUN is an exciting and stimulating book which "turns on" the reader. Its interest is enhanced by a list of 22 choice exercises given at the end, the last few of which were suggested by Conway. Exercise 19, for example, asks for a pseudo-number p such that $p + p \equiv *$. It is parenthetically stated that this is surprisingly difficult and leads to interesting subproblems. Of course once Alice and Bill will have discovered the application to games–which is mentioned in a short remark at the end of the book–and once they get used to handling them in the form of graphs (Figure 2), they will not experience too much difficulty with this problem.

Both books have much less than the average number of misprints. The reader of ONAG is perhaps kept slightly wondering by the fact that the relations $<$, $\lessdot$ as defined for games in the First Part are not proved consistent with such relations as defined for numbers (and games!–Chapter 1) in the Zeroth Part (and again on p. 78), nor is the necessity of such a proof indicated. It is, in fact, advantageous to give this consistency proof at a fairly early stage. Then game-theoretic considerations can be used to prove many properties of numbers as well as of games in a simple manner. The index suggests that Welter exists ("Welter, C. P., 153; his game, 153") but on p. 153 we find only "Welter's game" and a literary citation " ... And Welt'ring in his blood", which suggests that Welter is used figuratively. At any rate, reference [17] is not given. Reference 17 of ONAG is incomplete. At some places the discussion is a bit too terse for my taste, for example in the description of games with dynamic ties, where the requirement for a strategy of proper move selection, in addition to the marking of values on the vertices, is just barely hinted at. The chapter letterheads come without chapter numbers–perhaps because the editors noticed that at the beginning the class of numbers was void–which sometimes makes browsing and searching a bit awkward.

The late Dr. I. Breuer, a Jerusalem scholar and author of many books, once confided to me that he was much bothered by Ecclesiastes' strong discouragement of the writing of many books. He was further intrigued by the end of the verse which mentions excessive studiousness and tiredness of flesh. What is the connection to writing books? After pondering the problem he put forth the following explanation. Books which result from the tiredness of the flesh on which the author sits for hours without end because of excessive studiousness–those books my son, o beware of. But books which are inspired by the grey cells and elevate and please the spirit, those books it is pleasing to write; in fact, King Solomon after all did conceive himself Ecclesiastes, Proverbs and the Song of Songs![1]

ONAG in Hebrew means pleasure. Especially spiritual pleasure. But not always pure spiritual pleasure. The word allows the suggestion of pleasure triggered by subtle mixtures of spiritual and physical contentment, such as the SUN rays lighting and warming the world we live in.

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[1]Of course Bill's translation of the verse is already slanted towards this interpretation. He did not use one of the standard English translations, because they are uniformly bad and do not reflect the spirit of the original. In fact, Bill concluded that the common English Bible translations are even morphologically flawed. Otherwise, he reasoned, his creator Knuth would not have had to add two additional initials to Conway's name, because the readers of SUN could have been relied upon to know that Conway's own two initials are used for the divine name in many places in the Scriptures!

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Theory of modules, by Alexandru Solian, John Wiley & Sons Limited, London, New York, Sydney, Toronto, 1977, x + 420 pp., $26.50.

Somewhere between saying too little and saying too much lies good exposition. Most of the pitfalls are located to one side or the other of that rather narrow ridge where the essential ideas are provided without a deluge of trivialities. Being one who tends to fall off the ridge at regular intervals, it interests me to speculate on the reasons behind difficult lecturing or writing styles. One reason, of course, is inexperience, and I believe that criticism of exposition is an important part of graduate education. In seminar presentations, I feel that students are too often let off the hook because what they are doing is mathematically correct, even though what they are saying may be devastating for the understanding of the other participants. However teaching someone to teach is difficult, and perhaps dangerous too, if one is not absolutely sure of the difference between what enlightens and what confuses. Let us consider some of the other possible reasons behind incomprehensibility.

In my early years I was aware that I invariably understood some people and rarely understood others, without attributing this to any particular qualities of those involved. It was only later that I realized that those whom I could follow tended to be secure individuals, with enough self-confidence to tell me something I already knew, or remind me of something I knew a week ago. We are probably all a little sensitive to the reply, "But that's trivial," especially when it concerns something which we have found anything but trivial, and perhaps those who are least affected by the reply are by and large those who refrain from using it. When someone begins an explanation by assuming that his audience is plunged into the matter as deeply as he is, I usually feel that he is protecting himself from something. But of course insecurity is not always the reason for a bulldozer style. Sometimes it is a simple matter of insensitivity, an inability to realize that others are not