CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{sl}(2, \mathbb{C})$

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Let $\mathfrak{g}$ be a nonabelian Lie algebra over an algebraically closed field $K$ of characteristic 0. One is interested in the (algebraically) irreducible representations of $\mathfrak{g}$ acting on a vector space which is allowed to be infinite dimensional. The subject of enveloping algebras is largely concerned with these, but even in the simplest nonabelian case, with $\mathfrak{g} = \mathfrak{h}$ the 3-dimensional (nilpotent) Heisenberg algebra, as Dixmier remarks in discussing the situation when $K = \mathbb{C}$ in the preface to [2], "a deeper study reveals the existence of an enormous number of irreducible representations of $\mathfrak{h}$. . . . It seems that these representations defy classification. A similar phenomenon exists for $\mathfrak{g} = \mathfrak{sl}(2)$, and most certainly for all noncommutative Lie algebras."

However, as we shall see, the situation for $\mathfrak{h}$ and for $\mathfrak{sl}(2)$ turns out to be far nicer than hoped for. Indeed we announce here a determination and classification of all irreducible representations of $\mathfrak{h}$, of $\mathfrak{sl}(2)$, and of the 2-dimensional nonabelian Lie algebra, and thus of the prototypes respectively of nilpotent, simple, and solvable Lie algebras. As a guide to the meaning of "classification" and because our results use the same invariants, consider a classical situation of an (associative) algebra for which the irreducible representations have long been classified, namely, the algebra $B$ of formal linear differential operators with rational function coefficients, i.e., $B = K(q)[p]$, the (noncommutative) polynomials in an indeterminate $p$ where multiplication is determined by the relation $pq - qp = 1$. Then $B$ is a left principal ideal domain. Therefore [3] a $B$-module $M$ is simple if and only if $M \cong B/Bb$ for some $b \in B$ which is irreducible (i.e., $b = ac$ implies $a$ or $c$ is a unit); and $B/Bb \cong B/Ba$ if and only if $a$ and $b$ are similar, i.e., there exists $c \in B$ such that $(b, c) = 1$ and $a = [b, c]c^{-1}$ where $(b, c)$ is a
right g.c.d. and \([b, c]\) a left l.c.m. (which always exist) (similar is the noncommutative generalization of associate).

The subalgebra \(K[q] [p]\) of \(B\) generated by \(p, q\) is the Weyl algebra \(A_1\).

Since \(A_1 \cong U\mathfrak{h}/U\mathfrak{h}(x - \alpha)\) for \(0 \neq z \in \text{center } \mathfrak{h}\) and \(0 \neq \alpha \in K\), the problems of finding the irreducible representations for \(A_1\) and for \(\mathfrak{h}\) are equivalent. Our solution for this problem as well as for \(\mathfrak{sl}(2)\) involves the new notion of preserving, defined in terms of certain polynomials which we now introduce. For \(\alpha \in K\) let \(\mu_\alpha\) denote the valuation of \(K(q)\) determined by the prime \(q - \alpha\) of \(K[q]\), and extend \(\mu_\alpha\) to a function (also denoted \(\mu_\alpha\)) on \(B\) by setting \(\mu_\alpha(\Sigma_j b_j(q)p^j) = \min \{\mu_\alpha(b_j(q)) - j|j \geq 0\}\). Then define \(\theta_{\alpha, b}(\lambda) \in K[\lambda] (\alpha \in K, b = \Sigma_j b_j(q)p^j \in B)\) by

\[
\theta_{\alpha, b}(\lambda) = \sum_j ([((q - \alpha)^{-\mu_\alpha b - j} b_j(q))(\alpha)](-1)^j \lambda(\lambda + 1) \cdots (\lambda + j - 1).
\]

(It can be proved that \(\mu_\alpha\) is a valuation on \(B\), and extends to a valuation on the quotient division ring whose residue field is \(K(\lambda)\); then with \(\varphi_\alpha\) the corresponding place, \(\theta_{\alpha, b}(\lambda) = \varphi_\alpha((q - \alpha)^{-\mu_\alpha b} b)\).) Call \(b\) \(\alpha\)-preserving if \(\theta_{\alpha, b}(\lambda)\) has no positive integral root, and preserving it is \(\alpha\)-preserving for all \(\alpha \in K\). It can be shown that \(b\) is preserving if it is \(\alpha\)-preserving for a certain finite set of \(\alpha\)'s, in particular (when \(b\) is normalized to be in \(A_1\)) for the set of roots of the leading coefficient \(b_r(q)\); thus if \(K = C\) the property of \(b\) being preserving is computable given the roots of \(b_r(q)\).

The \(A_1\)-module \((K[p], q - \alpha)\) acts as \(-d/dp\) is simple and is precisely the simple \(A_1\)-module for which \(q\) has \(\alpha\) as an eigenvalue.

**Theorem 1.** If \(a \in B\) is irreducible and preserving then the \(A_1\)-module \(A_1/A_1 \cap Ba\) is simple.

**Theorem 2.** If \(M\) is a simple \(A_1\)-module then either \(M \cong (K[p], q - \alpha)\) acts as \(-d/dp\) for some \(\alpha \in K\) or \(M \cong A_1/A_1 \cap Ba\) for some \(a\) as in Theorem 1.

Since the \(A_1/A_1 \cap Ba\) above have no eigenvector for \(q\), the following completes the classification of the simple \(A_1\)-modules.

**Theorem 3.** Two simple \(A_1\)-modules \(A_1/A_1 \cap Ba, A_1/A_1 \cap Bb\) are isomorphic if and only if \(a\) and \(b\) are similar (in \(B\)).

Now consider the case of \(\mathfrak{g} = \mathfrak{sl}(2, K) = \mathfrak{h}\), with canonical basis \(e, f, h\). For \(\beta \in K\) the map \(e \rightarrow q, h \rightarrow 2qp - \beta, f \rightarrow -(qp - \beta)p\) extends to a homomorphism \(\rho_\beta\) of \(U\mathfrak{g}\) to \(B\). The simple \(\mathfrak{g}\)-modules for which \(e\) has an eigenvector \(\nu\) (with eigenvalue \(\alpha\)) are as follows: if \(\alpha = 0\), the highest weight modules \(L(\beta)\) \((\beta \in K)\) (with \(hv = \beta v\)); if \(\alpha \neq 0\), the simple Whittaker module \(W_\beta(\alpha)\) (see \([1], [4] : \alpha = \eta(e)\)), with basis \(t^0 = v, t^1, \ldots\) where \(ht^t = 2t^{t+1}, et^t = \alpha(t - 1)t^t, ft^t = \alpha^{-1}(t + 1)t(-t - t^2 + (\beta^2 + 2\beta)/4)\). The only isomorphisms among these
are \( Wh_\beta(\alpha) \cong Wh_\delta(\alpha) \) whenever \( \beta^2 + 2\beta = \delta^2 + 2\delta \). For any \( \beta \in K \) write \( \beta' \) for the other root of \( \lambda^2 + 2\lambda = \beta^2 + 2\beta \) i.e., \( \beta' = -\beta - 2 \).

**Theorem 4.** Suppose \( a \in U\mathfrak{g} \), \( \beta \in K \), \( \rho_\beta a \) is irreducible (in \( B \)) and \( \rho_\beta a \) and \( \rho_\beta a \) are preserving. Then the \( U\mathfrak{g} \)-module \( \rho_\beta U\mathfrak{g} / \rho_\beta U \mathfrak{g} \cap B(\rho_\beta a) \) is simple.

**Theorem 5.** If \( M \) is a simple \( U\mathfrak{g} \)-module then either \( M \cong L(\beta) \) for some \( \beta \in K \) or \( M \cong Wh_\beta(\alpha) \) for some \( \alpha, \beta \in K, \alpha \neq 0 \), or \( M \cong \rho_\beta U\mathfrak{g} / \rho_\beta U \mathfrak{g} \cap B(\rho_\beta a) \) for some \( a \) as in Theorem 4.

Again the following completes the classification.

**Theorem 6.** Two simple \( U\mathfrak{g} \)-modules \( \rho_\beta U\mathfrak{g} / \rho_\beta U \mathfrak{g} \cap B(\rho_\beta a), \rho_\delta U\mathfrak{g} / \rho_\delta U \mathfrak{g} \cap B(\rho_\delta b) \) are isomorphic if and only if \( \beta^2 + 2\beta = \delta^2 + 2\delta \) and \( \rho_\beta a \) and \( \rho_\delta b \) are similar (in \( B \)).

Analogous results hold for the 2-dimensional nonabelian Lie algebra, realized say as the subalgebra \( \mathfrak{b} = Kh + Ke \) of \( \mathfrak{g} \), with the following changes: the simple \( \mathfrak{b} \)-modules for which \( e \) has an eigenvector are \( Wh_\beta(\alpha) \) (for \( \alpha \neq 0 \)) and, for each \( \delta \in K, K\nu \subseteq L(\delta) \); restrict \( \beta \) to 0 and change the condition on preserving to the condition that \( \rho_\beta a \) be preserving and \( \theta_{\alpha,\rho_\beta a}(\lambda) \in K \) (or equivalently, \( a = e^u(ce + a) \) for some \( u \in N, c \in U\mathfrak{b} \) and \( 0 \neq a \in K \)).

The ring \( B \) is the localization of its subrings \( A_1 \) and \( \rho_\beta U\mathfrak{g} \) with respect to the multiplicative subset \( S = K[q] - \{0\} \).

**Theorem 7.** Every simple \( B \)-module \( N \) contains a unique simple \( A_1 \)-submodule \( \psi N \) and, for every \( \beta \in K \), a unique simple \( \rho_\beta U\mathfrak{g} \) submodule \( \psi_\beta N \); \( \psi N \) (resp. \( \psi_\beta N \)) is contained in every nonzero \( A_1 \)- (resp. \( \rho_\beta U\mathfrak{g} \)-)submodule of \( N \). Also \( B\mathfrak{n} \cong S^{-1}(\psi N) \cong S^{-1}(\psi_\beta N) \), and if \( M \) is a simple \( S \)-torsionfree \( A_1 \)- (resp. \( \rho_\beta U\mathfrak{g} \)-)module then \( \psi(S^{-1}M) \) (resp. \( \psi_\beta(S^{-1}M) \)) \( \cong M \). Thus the map \( N \rightarrow \psi N \) (resp. \( N \rightarrow \psi_\beta N \)) sets up a bijection between the set of isomorphism classes of simple \( B \)-modules and the set of isomorphism classes of \( S \)-torsionfree simple \( A_1 \)-modules (resp. \( U\mathfrak{g} \)modules with the Casimir element \( 4fe + h^2 + 2h \) acting as \( \beta^2 + 2\beta \)).

Here is a formula involving the \( \theta_{\alpha,\beta}(\lambda) \) which helps to explain their relevance to modules. If \( a \in A \), then for the action on \( (K[p], q - \alpha \) acts as \( -d/dp \), for every positive integer \( s \) we have

\[
(q - \alpha)^{-\mu a} a \cdot p^{s-1} = \theta_{\alpha,\beta}(s)p^{s-1} + \text{lower terms.}
\]

Somewhat similar formulas hold for the actions of \( U\mathfrak{g} \) on \( Wh_\gamma(\alpha) \) and \( L(\delta) \). The proof of Theorem 1 begins by showing that a maximal ideal \( J \) of \( A \) properly containing \( A \cap Ba \) intersects \( S \), and so \( q \) has an eigenvector on \( A/J \). Then one uses \( \alpha \)-preserving and (1). Theorem 4 is similar. The remaining theorems use properties of minimal annihilators and localizations. Theorems 2, 5 and 7 also depend on the following.
Lemma. If $b \in B$, there exists $d \in S$ such that $bd\,d^{-1}$ is preserving.

The proof of Theorem 7 also uses Theorem 1 and 4; if $N = B/Bb$ where $b$ is preserving then $\psi N = (A_1 + Bb)/Bb$.

References


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