SELFADJOINT OPERATOR EXTENSIONS
SATISFYING THE WEYL COMMUTATION RELATIONS

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ABSTRACT. Motivated by questions concerning uniqueness of unbounded derivations in commutative C*-algebras, and related problems on singular perturbations, we define two mixed global and infinitesimal versions of the Weyl operator commutation relations (one degree of freedom and infinite multiplicity), a weak one and a strong one. We announce two structure theorems of a geometric nature which characterize the nonselfadjoint symmetric operators entering in the Weyl systems. Proofs are only indicated.

Our starting point is the following variant of the Stone-von Neumann Uniqueness Theorem [12], [4b]. Let (U, V) be a pair of unitary one-parameter groups (always assumed strongly continuous) of operators on a separable Hilbert space H, and suppose that the Weyl commutation relation

\[ U(t)V(s) = V(s)U(t)e^{its} \quad \text{for all } s, t \in \mathbb{R} \]

holds. Then it is possible to represent the system in the form \( SU(t)S^{-1}f(x) = f(x + t), \) \( SV(s)S^{-1}f(x) = e^{isx}f(x) \), where \( S \) is an isometry of a space \( L^2(\mathbb{R}, N) \) of the norm-square integrable functions \( f \), with values in a separable Hilbert space \( N \), onto \( H \); the dimension of \( N \) being equal to the (uniform) multiplicity of the spectrum of \( U \).

Instead of (1) we consider the following infinitesimal Weyl relation with symmetric but generally nonselfadjoint generator. Let \( \{U(t)\}_{t \in \mathbb{R}} \) be a unitary one-parameter group on \( H \), and let \( Q \) be a symmetric operator with dense domain \( \mathcal{D}(Q) \) in \( H \). The corresponding relation

\[ (U(t)Qf, g) = (U(t)f, Qg) + \tau(U(t)f, g) \quad \text{for all } f, g \in \mathcal{D}(Q) \]

is here called the infinitesimal Weyl relation for the triple \( (U, Q, H) \). It is clearly equivalent to (1) if \( Q \) is essentially selfadjoint. But in scattering theory of singular perturbations, and in recent investigations of the author concerning uniqueness of unbounded derivations, the relation (2) for nonselfadjoint \( Q \) plays an interesting role. Simple examples show that the operator \( Q \) of a given system \( (U, Q, H) \) may

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2We refer to [LP] as a general reference, containing in addition important applications.

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have nonzero defect indices [4a] which can be equal or unequal. If they are equal, it may or may not be possible to extend $Q$ to a selfadjoint operator $\tilde{Q}$ so that the one-parameter group $V(s) = e^{is\tilde{Q}}$ is part of a (global) Weyl system $(U, V)$, i.e. such that (1) holds. The triples $(U, Q, H)$ which can be extended to global Weyl systems are called extendable.

In view of the negative examples alluded to above, it may be surprising that a certain canonical and minimal extension always exists.

**Theorem 1.** Let $(U, Q, H)$ be an infinitesimal Weyl system. Then symmetric extensions $Q_1$ of $Q$ exist such that

\[ D(Q_1) \text{ is invariant under } U(t) \text{ for all } t \in \mathbb{R}, \quad \text{and} \]

\[ Q_1 U(t)f = U(t)Q_1 f - t U(t)f \text{ for all } f \in D(Q_1). \]

There is a unique smallest symmetric extension $Q_m$ satisfying (3), i.e. $Q_m \subseteq Q_1$ for all symmetric extensions $Q_1$ satisfying (3). The operator closure $\tilde{Q}_m$ satisfies (3) as well and is the unique smallest symmetric and closed extension of $Q$ satisfying (3).

Here is an easy

**Corollary 2.** If the system $(U, Q, H)$ is extendable to a global Weyl system $(U, V)$ with $V(s) = e^{is\tilde{Q}}$ then $Q_m \subseteq \tilde{Q}$.

**Definition.** We say that the system $(U, Q, H)$ has $U$-indices $(p, q)$ if the minimal operator $Q_m$ has defect indices $(p, q)$. It can readily be shown that the $U$-indices are equal whenever $Q$ has equal defect indices, but in general the indices of $Q$ may be infinite while those of $Q_m$ are finite. Theorem 4 below is a converse to Corollary 2.

**Theorem 3.** Let $(U, Q, H)$ be an infinitesimal Weyl system with at least one finite $U$-index. Then there is a Hilbert space $K$ containing $H$, and a global Weyl system $(U, V)$ on $H$ such that $U$ reduces to $U$ on $H$, $H$ is semi-invariant [9] for $V$, and $Q$ is contained in the infinitesimal generator for the contraction semigroup $s \rightarrow P_H V(s)|_H$.

A structure theory for the associated contraction semigroups, due to P. Muhly, will appear in a joint article with the author [2b].

The next result concerns the relation (3) for extendable systems. Hence it is stated for operators in $L^2$-spaces, and $U(t)$ is translation. Vanishing conditions of the Fourier transform on a set of measure zero clearly give rise to systems (3). The theorem is a partial converse to this statement.

**Theorem 4.** Let $Q_1$ be a symmetric and closed operator in $L^2(\mathbb{R})$ which is contained in the multiplication operator $\tilde{Q}h(x) = xh(x)$. Assume that $D(Q_1)$ is translation invariant, and that $D(\tilde{Q}Q_1)$ is a core for $Q_1$. Then (3) holds and
the closed set \( \Lambda = \{ \lambda \in \mathbb{R} | \hat{h}(\lambda) = 0 \text{ for } \forall h \in \mathcal{D}(Q_1) \} \) is of zero measure. Moreover \( \mathcal{D}(Q_1) = \{ h \in \mathcal{D}(\hat{Q}) | \hat{h}(\lambda) = 0 \text{ for } \forall \lambda \in \Lambda \} \).

The core condition can be omitted in the following cases: (1) \( \Lambda \) is a Cantor set, or (2) \( Q_1 \) has finite defect. In general it can be slightly weakened, but not omitted. In case (2) the result generalizes to arbitrary multiplicity.

The proofs of results 1 through 3 involve general operator theory [1], including theorems of Phillips [5] and Naimark [3], while the proof of Theorem 4 is based on a function theoretic approach to the extension theory of [4a] made possible by the extendability assumption and the Stone-von Neumann Theorem. (The function theory is based on Wiener-Tauberian consideration.)

Generalizations to group representations [2a], [6], [7], [8], [11] and field theory [10] would be of interest. The possibility of removing the finiteness assumption in Theorem 3 is related via the proof to a well-known conjecture of Phillips [5]. We conjecture the conclusion of Theorem 4, also without the finiteness assumption on the \( U \)-index, but it seems hard to settle either of the conjectures. It appears difficult in the general case, for two different extensions to establish the existence of a wave operator similarity which commutes with \( U \).

However the extendability question has an answer in full generality, i.e. no restriction on the indices. The proof uses arguments that naturally extend [4a].

**Theorem 5.** Let \((U, Q, H)\) be an infinitesimal Weyl system, and let \( P_{\pm} \) denote the orthogonal projections onto the respective defect spaces \( \mathcal{D}_\pm \) for \( Q_m \). The system is extendable if and only if there exists a partial isometry \( S \) of \( \mathcal{D}_+ \) onto \( \mathcal{D}_- \) such that \( P_-(2[S, U(t)] - i(t + S)U(t)(I + S))P_+ = 0 \) for all \( t \in \mathbb{R} \). Here \([\cdot, \cdot]\) denotes the commutator bracket.

If it is assumed in addition that the spectral measure \( dE_\lambda \) of \( U \) is absolutely continuous, then extendability is equivalent to the validity of the following identity

\[
P_-(2[S, E_\lambda] - (I + S)D_\lambda(I + S))P_+ = 0 \quad (\lambda \in \mathbb{R})
\]

for some partial isometry \( S \) of \( \mathcal{D}_+ \) onto \( \mathcal{D}_- \). Here \( D_\lambda \) denotes the Radon-Nikodym derivative of \( dE_\lambda \).

We finally point out that Theorem 5 has a complete generalization to the case when the \( W^* \) algebra generated by the spectral projections of \( U \) is replaced by an arbitrary noncommutative \( W^* \)-algebra, and when the map \( U(t) \rightarrow it \) \( U(t) \) is replaced by a spatial derivation which is implemented by a symmetric nonself-adjoint operator. This answers a question raised in a recent article of the author.

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**REFERENCES**


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2. P. E. T. Jørgensen (a) (with R. T. Moore), *Commutation relations for operators, semigroups, and resolvents in mathematical physics and group representations* (preprint); (b) (with P. S. Muhly), *Selfadjoint extensions satisfying the Weyl operator commutation relations* (in preparation).


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