FINITENESS THEOREMS FOR POLYCYCLIC GROUPS
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Introduction. A group $G$ is polycyclic if it is built up from the identity by finitely many successive extensions with cyclic groups, or equivalently if it is isomorphic to a soluble group of matrices over $\mathbb{Z}$ (not obvious!). The second definition makes it clear that the normal subgroups of finite index in $G$ intersect in $1$, so one may hope that the finite quotient groups of $G$ will carry a lot of information about the structure of $G$. The first main result says that in fact they “almost” determine $G$ up to isomorphism, i.e. they do so up to finitely many possibilities. (Examples show that there really are finitely many possibilities, not just one.) The second main result is a sort of “concrete” analogue of this: if $G$ is contained in $GL_n(\mathbb{Z})$, then there are only finitely many possibilities up to conjugacy in $GL_n(\mathbb{Z})$ for subgroups $H$ in $GL_n(\mathbb{Z})$ such that $H$ is “conjugate to $G$ modulo $m$” for all nonzero integers $m$. This is related to classical results in arithmetic, like the fact that there are only finitely many inequivalent integral quadratic forms with given determinant, and the Hasse-Minkowski Theorem.

Results. Denote by $F(G)$ the set of isomorphism classes of finite quotients of a group $G$, and by $\hat{G}$ the profinite completion of $G$. For polycyclic-by-finite groups $G$ and $H$, $F(G) = \hat{F}(H)$ if and only if $G \cong \hat{H}$; if this holds we say that $G$ and $H$ belong to the same $\sim$-class.

THEOREM 1. Every $\sim$-class of polycyclic-by-finite groups is the union of finitely many isomorphism classes.

A major ingredient in the proof of this is a result about arithmetic groups. Let $G$ be an algebraic matrix group defined over $\mathbb{Q}$, and denote by $\pi_m$: $G(\mathbb{Z}) \rightarrow G(\mathbb{Z}/m\mathbb{Z})$ the canonical map. For subgroups $X$ and $Y$ of $G(\mathbb{Z})$, say $X \sim_G Y$ if for every $m \neq 0$, $X\pi_m$ and $Y\pi_m$ are conjugate in $G(\mathbb{Z}/m\mathbb{Z})$. 

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**Theorem 2** (F. J. G. and D. S.). Every $\sim_G$-class of soluble-by-finite subgroups of $G(\mathbb{Z})$ is the union of finitely many conjugacy classes in $G(\mathbb{Z})$.

Special cases of Theorem 1 have appeared in [P1], [P2], [GS1], and a special case of Theorem 2 in [GS1].

**Auxiliary results.** We state now some further results used in the proofs.

**Theorem 3** [S]. If $G$ is a polycyclic group and $d$ is a positive integer, then up to isomorphism there are only finitely many extensions of $G$ by a group of order $d$.

This is needed for Theorem 1. The next three results are needed for Theorem 2.

**Theorem 4** [S]. If $G$ is an algebraic matrix group defined over $\mathbb{Q}$ and $X$ is a soluble subgroup of $G(\mathbb{Z})$, then the subgroups of $G(\mathbb{Z})$ which contain $X$ as a normal subgroup of finite index lie in finitely many conjugacy classes in $G(\mathbb{Z})$.

Let $G \leq GL_n$ be an algebraic matrix group defined over $\mathbb{Q}$, and let $\pi_m$ now denote the canonical map $\mathbb{Z}^n \to (\mathbb{Z}/m\mathbb{Z})^n$. For $\mathbb{Z}$-submodules $A$ and $B$ of $\mathbb{Z}^n$, say $A \sim_G B$ if for every $m \neq 0$, $A\pi_m$ and $B\pi_m$ lie in the same orbit of $G(\mathbb{Z}/m\mathbb{Z})$.

**Theorem 5** [GS2]. Every $\sim_G$-class of $\mathbb{Z}$-submodules of $\mathbb{Z}^n$ is the union of finitely many orbits of $G(\mathbb{Z})$.

For the next result, let $\mathfrak{o}$ be the ring of integers in an algebraic number field and denote by $\mathcal{P}$ the set of all nonzero prime ideals of $\mathfrak{o}$. Call a subset $Q$ of $\mathcal{P}$ ample if every subgroup of finite index in the units group $\mathfrak{o}^*$ of $\mathfrak{o}$ contains a subgroup of the form $(1 + \mathfrak{a}) \cap \mathfrak{o}^*$ where $\mathfrak{a}$ is an intersection of members of $Q$.

**Theorem 6** [GS3]. If $F$ is a finite subset of $\mathcal{P}$ and $\mathcal{P} - F$ is partitioned into finitely many subsets, then at least one of these subsets is ample.

**Outline proof of Theorem 1.** Consider a set $C$ of polycyclic-by-finite groups, contained in a single $^\sim$-class. By Theorem 3 we may assume that for each $G \in C$, the Fitting subgroup $N_G$ of $G$ is torsion-free and $G/N_G$ is free abelian. Since [P3] $\hat{N}_G$ is the Fitting subgroup of $\hat{G}$, we may apply the special case of Theorem 1 for nilpotent groups [P1] and assume that the groups $N_G$ for $G \in C$ are all isomorphic. We then use Theorem 2, applied to the arithmetic group $\text{Aut}(N_G)$, to reduce to the case where the action of $G$ on $N_G$ is the same for all $G \in C$; i.e. the pairs $(G/\beta_1(N_G), N_G)$ are all isomorphic. Write $Q_G$ for the hypercentre of $G$. Using a cohomological result due to Robinson [R] one can further reduce to the situation where the groups $G/Q_G$ are all isomorphic, compatibly with the isomorphisms linking the various $N_G$. 

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The idea is now to form semisimple splittings of the groups in $\mathcal{C}$ (see [T]). For each $G$ we construct an abelian subgroup $T_G \leq \text{Aut}(G)$ such that the split extension $G \rtimes T_G$ is equal to $M_G \rtimes T_G$ where $M_G$ is the Fitting subgroup of $G \rtimes T_G$. Then $G \rtimes T_G$ can be embedded in a well-known way into some $GL_n(Z)$, by making $M_G$ and $T_G$ act on a suitable factor ring of the group ring $ZM_G$. If the groups $T_G$ are defined in a sufficiently uniform manner, we can arrange that an isomorphism from $H$ to $G$ induces an isomorphism from $(H \rtimes T_H)\wedge$ to $(G \rtimes T_G)\wedge$ sending $T_H$ to $T_G$. In this situation it is not hard to deduce that $G \sim_{GL_n} H$ in $GL_n(Z)$. A second application of Theorem 2 completes the proof.

Our construction of semisimple splittings differs from those in the literature. Roughly speaking, we construct a certain canonical family $\mathcal{X}(G)$ of nilpotent supplements for $N_G$ in $G$, $G$ being any polycyclic group. To get the required "uniformity", we then choose $C_G/Q_G \in \mathcal{X}(G/Q_G)$ simultaneously for all $G \in \mathcal{C}$, using the isomorphisms between the groups $G/Q_G$. The group $T_G$ is defined to act trivially on $C_G$ and to act like the Jordan semisimple component of $\text{Inn}(C_G)|_{N_G}$ on $N_G$; the existence of such a $T_G$ is a direct consequence of the fact that $C_G$ is nilpotent.

**Outline proof of Theorem 2.** Consider an algebraic $Q$-group $G$ and a set $\mathcal{C}$ of soluble-by-finite subgroups of $G(Z)$, contained in a single $\sim_G$-class. Using Theorem 4 and induction on derived length, we may assume that $\mathcal{C}$ consists of abelian groups. Now consider some special cases. If $\mathcal{C}$ consists of unipotent groups, the result is an easy consequence of Theorem 5. If $\mathcal{C}$ consists of abelian $d$-groups and $G = GL_n$, the result is proved with the help of Theorem 6: suppose $X \sim_G Y$ in $GL_n(Z)$; we diagonalize $X$ and $Y$ over some ring $\mathcal{O}$ of algebraic integers, and then for each prime $\mathfrak{p}$ of $\mathcal{O}$ we find a permutation matrix $\sigma(\mathfrak{p})$ such that $Y \equiv X^{\sigma(\mathfrak{p})} \mod \mathfrak{p}$. There exists a permutation matrix $\tau$ such that $\{\mathfrak{p} | \sigma(\mathfrak{p}) = \tau\}$ is ample, and one can deduce that then $Y = X^\tau$. To deduce the result for abelian $d$-subgroups in a general $G$ one uses the conjugacy of maximal tori.

A major part of the proof consists in reducing the problem to the special cases mentioned; this involves a series of rather complicated arguments which we cannot go into here. At several points in the proof, and particularly in the proof of Theorem 5, a key role is played by finiteness theorems of Borel [B] and Borel-Serre [BS].

**References**


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