THE DUALITY OPERATION IN THE CHARACTER RING
OF A FINITE CHEVALLEY GROUP
BY DEAN ALVIS

It is possible (as in [4]) to define a duality operation \( \xi \mapsto \xi^* \) in the ring of virtual characters of an arbitrary finite group with a split \((B, N)\)-pair of characteristic \( p \). Such a group arises as the fixed points under a Frobenius map of a connected reductive algebraic group, defined over a finite field [1]. This paper contains statements of several general properties of the duality map \( \xi \mapsto \xi^* \) and two related operations (see §§2 and 4). The duality map \( \xi \mapsto \xi^* \) generalizes the construction in [2] of the Steinberg character, and interacts well with the organization of the characters from the point of view of cuspidal characters (§6). It is hoped that there is also a useful interaction with the Deligne-Lusztig virtual characters \( R^G_T \). Partial results have been obtained in this direction (§5). Detailed proofs will appear elsewhere.

1. Let \( G \) be a finite group with split \((B, N)\)-pair of characteristic \( p \). Let \( (W, R) \) be the Coxeter system, and let \( P_j = L_j V_j \) be the standard parabolic subgroup corresponding to \( J \subset R \), with \( V_j = O_p(P_j) \) (see [3] for definitions and notations). Let \( \text{char}(G) \) denote the ring of virtual characters of \( G \), and \( \text{Irr}(G) \) the set of irreducible characters of \( G \), all taken in the complex field. For \( J \subset R \) and \( \xi \in \text{char}(G) \) define

\[
\xi_{(P_j/V_j)} = \Sigma(\xi, \tilde{\lambda}^G)_{G^\lambda}
\]

where \( \sim \) denotes extension to \( P_j \) via the projection \( P_j \rightarrow L_j \cong P_j/V_j \), and the sum is over all \( \lambda \in \text{Irr}(L_j) \). Let \( \xi_{(P_j)} = \xi_{(P_j/V_j)} \). The duality map is then defined by:

1.2 DEFINITION. \( \xi^* = \Sigma_{J \subset R} (-1)^{|J|} \xi_{(P_j)}^G \), for all \( \xi \in \text{char}(G) \).

2. The truncation map \( \xi \mapsto \xi_{(P_j/V_j)} \) and the map \( \lambda \mapsto \tilde{\lambda}^G \) behave in much the same way as ordinary restriction and induction. The following basic properties follow directly from the structure theorems [3].

2.1 FROBENIUS RECIPROCITY. Let \( \xi \in \text{char}(G) \) and \( \lambda \in \text{char}(L_j) \). Then...
\((\xi, \tilde{\lambda}^G)_G = (\xi_{(P_J)}, \tilde{\lambda})_{P_J} = (\xi_{P_J/V_J}, \lambda)_L\).

2.2 Transitivity. If \(K \subseteq J \subseteq R\), let \(Q_K\) be the standard parabolic subgroup \(P_K \cap L_J\) of \(L_J\) and let \(V_{J,K} = O_p(Q_K) = L_J \cap V_K\). Then if \(\xi \in \text{char}(G)\) and \(\xi \in \text{char}(L_J)\), we have

\[ (\xi_{P_J/V_J})_{Q_K/V_J,K} = \xi_{(P_K/V_K)} \]

and

\[ (\tilde{\lambda}^L)_{G} = \tilde{\lambda}^G. \]

2.3 Intertwining number theorem. Let \(\lambda_i \in \text{char}(L_i)\) for \(i = 1, 2\).

Then

\[ (\tilde{\lambda}^G_1, \tilde{\lambda}^G_2)_G = \sum_{w \in W_{J_1,J_2}} (\lambda_1(Q_{K_1}/V_{J_1}), w^\lambda_2(Q_{K_2}/V_{J_2,K_2}))_{L_{K_1}} \]

where \(W_{J_1,J_2}\) is the set of distinguished \(W_{J_1} - W_{J_2}\) double coset representatives, \(W_{K_1} = W_{J_1} \cap wW_{J_2}\) and \(W_{K_2} = W_{J_2} \cap w^{-1}W_{J_1}\).

2.4 Subgroup theorem. Let \(\lambda \in \text{char}(L_{J_1})\). Then

\[ (\tilde{\lambda}^G)_{P_{J_2}/V_{J_2}} = \sum_{w \in W_{J_1,J_2}} w^{-1}(\lambda(Q_{K_1}/V_{J_1,K_1}))_{L_{J_2}}. \]

Here \(K_1\) is as in 2.3 (note: \(w^{-1}L_{K_1} = L_{K_2}\)).

3. The results of this section are of independent interest, and are due to Curtis ([4]). They are needed to apply the results of §2 to the duality operation.

3.1. Lemma. Let \(w \in W, \ wL_{J_2} = L_{J_1}, \ w\lambda_2 = \lambda_1, \ where \ \lambda_i \in \text{char}(L_{J_i}).\) Then \(\lambda^G_1 = \lambda^G_2.\)

The idea of the proof is to show that the numbers \((\tilde{\lambda}^G_1, \tilde{\lambda}^G_2)_G\) are all the same for \(i, j = 1, 2\). The proof in [3] (for the special case when \(\lambda_1, \lambda_2\) are cuspidal) can be modified to work in the present situation.

The following is Lemma 2.5 of [4].

3.2. Lemma. Let \(a_{J_2,J_1,K} = |\{w \in W_{J_1,J_2} | W_K = W_{J_1} \cap wW_{J_2}\}|.\) Then

\[ \sum_{J_2 \subseteq R} (-1)^{|J_2|} a_{J_2,J_1,K} = (-1)^{|K|}. \]
4. The first main result relates duality and the operations $\xi \rightarrow \xi_{(P_j/V_j)}$ and $\lambda \rightarrow \lambda^G$. Part (1) is Theorem 1.3 of [4].

Theorem. (1) $(\xi_{(P_j/V_j)})^* = (\xi_{(P_j/V_j)})^*$ for $J \subseteq R$, $\xi \in \text{char}(G)$
(2) $(\lambda^G)^* = (\lambda^G)^*$ for $J \subseteq R$, $\lambda \in \text{char}(L_j)$.

We provide a sketch of the proof of (2). Let $J_1 = J$. Using 2.4, 2.2, and then Lemma 3.1 (noting that $L_{K_1} = W_{L_{K_2}}$ by Proposition 2.6 of [3]) we have

$$\lambda^G)^* = \sum_{J_2 \subseteq R} (-1)^{|J_2|} \sum_{w \in W_{J_1}, J_2} \lambda(Q_{K_1/V_{J_1}, K_1)^*}$$

The proof is then completed by applying Lemma 3.2 and 2.2.

4.2 Theorem. The map $\xi \rightarrow \xi^*$, from $\text{char}(G) \rightarrow \text{char}(G)$ is an isometry of order two. In particular, $\xi^{**} = \xi$ and $\pm \xi^* \in \text{Irr}(G)$, whenever $\xi \in \text{Irr}(G)$.

In order to prove Theorem 4.2, one first proves that $(\xi_1, \xi_2)_G = (\xi_1^*, \xi_2^*)_G$. It then suffices to prove $\xi^{**} = \xi$. The key is to apply Theorem 4.1 part (1) to the expression for $\xi^{**}$. We have

$$\xi^{**} = \sum_{J \subseteq R} (-1)^{|J|} \xi_{(P_j/V_j)}^* \sim^G$$

$$= \sum_{J \subseteq R} (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|} \xi_{(P_K)}^G$$

using 2.2. To finish the proof, note that $\sum (-1)^{|J|}$ summed over all $J$ such that $K \subseteq J \subseteq R$ is zero unless $K = R$.

5. It is clear that $\xi^* = (-1)^{|J|} \xi$ for any cuspidal $\xi \in \text{Irr}(G)$. Thus by applying Theorem 4.1 part (2) we have:

5.1 Corollary. Let $\lambda \in \text{Irr}(L_{\lambda})$ be cuspidal. Then $(\lambda^G)^* = (-1)^{|J|} \lambda^G$.

Thus duality permutes (up to sign) the components of $\lambda^G$. We can thus determine the “sign” of $\xi^*$ as follows: $(-1)^{|J|} \xi^*$ is in $\text{Irr}(G)$ if $\xi \in \text{Irr}(G)$ is a component of $\lambda^G$, $\lambda \in \text{Irr}(L_j)$ cuspidal. In particular, $\xi \rightarrow \xi^*$ permutes the principal series characters, i.e. the components of $\lambda^G$, $\lambda \in \text{Irr}(L_j)$. A more explicit result is known for the components $\xi_{\varphi, q}$ of $1^G_{\text{B}(q)}$ in a system of groups $\{G(q)\}$ of type $(W, R)$. Specifically, $\xi_{\varphi, q}^* = \xi_{e\varphi, q}$ where $e$ is the sign character of $W([4])$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Finally, consider the case $G = G^F$ where $G$ is a reductive algebraic group and $F: G \to G$ is a Frobenius map over $F_q$. Let $R^G_T \theta$ denote the Deligne-Lusztig generalized character of $G$ (an $F$-stable maximal torus of $G$, $\theta$ a linear character of $T^F$). It is natural to ask whether

\[(R^G_T \theta)^* = \pm R^G_T \theta\]

holds. The following suggests the answer is yes.

\[(R^G_T \theta)^*(s) = \pm R^G_T \theta(s)\]

for semisimple elements $s$ of $G$. The $\pm$ sign in 5.3 does not depend on the particular element $s$ of $G$. The proof of 5.3 uses several results of [5]. (Note added in proof: The conjecture 5.2 has been proved by G. Lusztig.)

5.4 Example. Let $G = G^F$ as above, with (relative) Coxeter system $(W, R)$. Let $V$ be the set of unipotent elements of $G$ and let $\epsilon_V$ be the characteristic function of $V$. A recent result of Springer (Theorem 1 of [6])\(^1\) shows

$$\epsilon_V = q^d \sum_{J \subset R} (-1)^{|J|} |P_J|^{-1} \chi^G_{V_J}$$

where $d = \dim(G/B)$, $B$ a Borel subgroup of $G$. Applying Theorems 4.1 and 4.2 we have:

5.5 Theorem. (1) $\epsilon^*_V = (q^d/|G|) \rho_G$ where $\rho_G$ is the regular character of $G$.

(2) For $\xi \in \text{Irr}(G)$,

$$\frac{1}{\xi(1)} \sum_{v \in V} \xi(v) = q^d (\xi^*(1)/\xi(1)).$$

(3) For $\xi \in \text{Irr}(G)$, $|\xi^*(1)|_{p'} = \xi(1)_p$, where $p$ is the characteristic of $F_q$ and $n_p'$ is the $p'$ part of $n$.

(4) For $\xi \in \text{Irr}(G)$, $1/\xi(1) \sum_{v \in V} \xi(v)$ is, up to sign, a power of $p$.

Part (4) of Theorem 5.5 confirms a special case of a conjecture of Macdonald (see [6]), namely the case when $q = p$ is prime.

REFERENCES


\(^1\)The author is indebted to T. A. Springer for communicating both his results in [6] and the suggestion of G. Lusztig of combining them with duality.


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403