

have uncountably many ergodic equilibrium states. Fortunately, all is not as bad as that. For one thing, much of the pathology disappears if one restricts oneself to the smaller space  $\mathfrak{B}$  of interactions. For another thing, there is the classic theorem of Mazur, according to which the set of points  $\Phi$  at which  $P$  has a unique tangent is a dense  $G_\delta$ -set in  $\mathfrak{B}$ , so from the point of view of Baire-category the occurrence of more than one ergodic equilibrium state for the same interaction is an exceptional phenomenon. The last chapter of our book discusses a strengthening of this statement. Think of some finite dimensional subspace  $\mathcal{N}$  of  $\mathfrak{B}$ . If  $\mathcal{N}$  is well behaved, one expects the majority of points  $\Phi \in \mathcal{N}$  to possess exactly one equilibrium state. Exceptionally, there may be two ergodic equilibrium states, but the subset of  $\mathcal{N}$  where this is the case should be small. Even smaller should be the subset of  $\mathcal{N}$  where there are three, and so on. Let us say that the Gibbs Phase Rule holds in  $\mathcal{N}$  if the set  $\{\Phi \in \mathcal{N} : \Phi \text{ has } k \text{ ergodic equilibrium states}\}$  has Hausdorff-dimension at most  $n - k + 1$  in  $\mathcal{N}$ , where  $n = \dim(\mathcal{N})$ . The set of  $n$ -dimensional subspaces of  $\mathfrak{B}$  can be made into a complete metric space  $\mathfrak{G}_n$ . The precise version of the statement "The Gibbs Phase Rule holds generically" is the theorem that  $\{\mathcal{N} \in \mathfrak{G}_n : \text{The Gibbs Phase Rule holds in } \mathcal{N}\}$  is a dense  $G_\delta$ -set in  $\mathfrak{G}_n$ . This is quite satisfactory, although conceivably one might want to know more, for instance, if  $\dim(\mathcal{N}) = 2$  and  $k = 3$  ("triple point" or coexistence of three pure phases), one expects only isolated points in  $\mathcal{N}$ , not merely a set of Hausdorff dimension 0; similarly for  $k = 2$  (coexistence of two pure phases) one should have nice curves in some sense or other. Perhaps future research will succeed in this direction.

One more remark on the contents of the book. All definitions and theorems have their quantum mechanical analogues. In fact, the material is so organized that the two cases, classical and quantum, are discussed side by side, so that the investigation proceeds in parallel. In fact, on the level of general theory, there is hardly any difference, and the effect of quantum modification is present only for specific properties of specific models.

My own assessment is that this book is a valuable compendium for research workers in the mathematical aspects of statistical mechanics, and it should also succeed in attracting outsiders from the mathematical community to acquaint themselves with a fascinating topic.

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*Thermodynamic formalism: The mathematical structures of classical equilibrium statistical mechanics*, by David Ruelle, Encyclopedia of Mathematics and its Applications (Gian-Carlo Rota, Editor), vol. 5, Addison-Wesley, Reading, Mass., 1978, xix + 183 pp., \$21.50.

The book under review is concerned with the general aspects of classical equilibrium statistical mechanics of lattice systems and some generalizations. Before commenting on the book we shall describe some of the main *mathematical* issues which arise in this deep and active area of mathematical

physics. Physical motivation will be discussed only occasionally. There is a brilliant discussion of this as well as the historical background of the statistical mechanical formalism in the lengthy introduction by A. S. Wightman to [1].

The purpose of equilibrium statistical mechanics is to describe the (equilibrium) thermodynamic properties of certain “large” physical systems in terms of the *microscopic laws* of physics. The striking thing is that typically the values of  $n = 2$  or 3 observables (almost) determine a thermodynamic state while  $N \cong 10^{23}$  parameters determine a microscopic state. One usually is interested in approximately 5 to 10 thermodynamic (= macroscopic) observables and the relations between them. {These are often called equations of state.} Typically the values of a small subset of these observables will (almost) determine a state and there will be a distinguished function  $G$  of the values of this basic set such that (almost all of) the values of all other thermodynamic observables in this state will be obtained by evaluating appropriate derivatives of  $G$  at this state value. Such a function is called a *thermodynamic potential*. In order to agree with the general laws of thermodynamics it will have certain *convexity* properties. In the case of a simple fluid the (absolute) temperature  $T$  and chemical potential  $\mu$  may serve as state parameters and the pressure  $P$  as the thermodynamic potential. In the case of a ferromagnet the temperature  $T$  and the magnetic field  $H$  may serve as state parameters and the free-energy per site  $f$  as the thermodynamic potential. More explicitly now, the problem of statistical mechanics is to obtain the appropriate thermodynamic potential with appropriate convexity properties from the microscopic laws of physics.

Note that there was some waffling above about states when we used the word “almost”. This was because for certain special values of the state parameters *phase transitions* can occur i.e. some of the values of certain other thermodynamic observables are not uniquely determined. At such points the state structure gets more complicated and interesting! For example in the case of the Lenz-Ising ferromagnet (of at least 2 dimensions), spontaneous magnetization (per site) can occur when  $T$  is sufficiently small and  $H = 0$  and this leads to an ambiguity in the value of magnetization at these state parameter values. It is a major challenge of statistical mechanics to predict and explain phase transitions.

An important feature of thermodynamic systems is that the properties we are interested in are independent of shape and size at least when we are looking at intensive properties such as pressure, temperature, density, free-energy per site (and not total energy, volume, etc.). This will allow us to let the statistical mechanical model of our system to become “infinitely large.” This is called “taking the thermodynamic limit (T.L.)” and it is required to explain phase transitions. This is the *thermodynamic formalism* referred to in the title of the book.

We shall now discuss a typical lattice system, the Lenz-Ising model for ferromagnetism, and indicate some of the key issues to be resolved. The model is viewed as a collection of particles with spin on the lattice  $\mathbf{Z}^{\nu}$ ,  $\nu = 1, 2, 3$ , where an element of  $\mathbf{Z}^{\nu}$  is called a *site*. A particle  $\{i\}$  can exist in one of two states  $\sigma_i$ :  $\sigma_i = +1$  if the spin is up and  $\sigma_i = -1$  if the spin is down. Let

$\Omega = \prod_{i \in \mathbf{Z}^p} \{+1, -1\}$  which we view as a compact Hausdorff space when  $\{+1, -1\}$  is given the discrete topology. This is the set of possible configurations  $\{ = \text{microscopic states} \}$  of the system. A typical element is denoted by  $\sigma = \{\sigma_i\}$  where  $\sigma_i$  is the spin-state at site  $\{i\}$ . A *state* is a probability measure on  $\Omega$  and elements of  $\mathcal{C}(\Omega)$ , the algebra of real-valued continuous functions on  $\Omega$ , are called *observables*.  $\mathbf{Z}^p$  acts naturally as a group of homeomorphisms on  $\Omega$  and one can discuss the  $\mathbf{Z}^p$ -invariant states on  $\Omega$ . We denote this convex set by  $I$ .

Let  $J$  be a *positive* real number. {This is the ferromagnetic condition on the “exchange interaction.”} For each finite region  $\Lambda \subset \mathbf{Z}^p$ , we define the *interaction Hamiltonian*, which is a function from  $\Omega_\Lambda$  to  $\mathbf{R}$ , as follows:

$$\mathcal{H}_\Lambda\{\sigma_\Lambda\} =: - \sum_{\substack{i, j \in \Lambda \\ \|i-j\|=1}} J\sigma_i\sigma_j - H \sum_{i \in \Lambda} \sigma_i$$

where  $\sigma_\Lambda \in \Omega_\Lambda =: \prod_{i \in \Lambda} \{+1, -1\}$  and  $H \in \mathbf{R}$  is the magnitude of a homogeneous magnetic field applied to the system.  $\mathcal{H}_\Lambda\{\sigma_\Lambda\}$  is the local energy of a configuration  $\sigma$  such that  $\sigma|_\Lambda = \sigma_\Lambda$ . Since nature seems to prefer states which minimize energy, the above suggests spin alignment up is preferred if  $H \gg 0$  and spin alignment down is preferred if  $H \ll 0$ , but if  $H = 0$  nature only prefers alignment, up or down! Roughly this ambiguity is the intuitive reason behind the phase transitions observed in ferromagnets for sufficiently low temperature and zero magnetic field. In any case the family of interaction Hamiltonian  $\{\mathcal{H}_\Lambda\}$  is the microscopic data out of which the thermodynamic properties of the system must be deduced.

Given the family  $\{\mathcal{H}_\Lambda\}$  and a value  $T > 0$  of absolute temperature, we define the corresponding family of *partition functions*  $\{Z_\Lambda(T, H)\}$  as follows:

$$Z_\Lambda(T, H) =: \sum_{\sigma_\Lambda \in \Omega_\Lambda} \exp[-\beta \mathcal{H}_\Lambda\{\sigma_\Lambda\}]$$

where  $\beta = 1/kT$  and  $k =$  the Boltzmann constant. The partition functions contain all of the thermodynamic information for the system. In particular they are used to define the following families of thermodynamic objects:

(1) The *Gibbs ensemble* on  $\Omega_\Lambda$  which is a probability measure on  $\Omega_\Lambda$  indexed by  $T$  and  $H$  and is defined by:

$$\gamma_\Lambda(T, H)\{\sigma_\Lambda\} =: Z_\Lambda^{-1}(T, H) e^{-\beta \mathcal{H}_\Lambda\{\sigma_\Lambda\}};$$

(2) the *free-energy per site* which is defined by:

$$f_\Lambda(T, H) =: -\ln Z_\Lambda(T, H) / \beta|\Lambda|$$

where  $|\Lambda| =$  the number of sites in  $\Lambda$ .

Skipping over the technicalities of what the precise meaning of “ $\lim_{\Lambda \rightarrow \mathbf{Z}^p}$ ” is, the main mathematical problems for our system are:

(A) Describe the measures on  $\Omega$  obtained as thermodynamic limits of the  $\{\gamma_\Lambda(T, H)\}$  i.e.  $\gamma$  is such a measure if there exists a sequence  $\Lambda_n \rightarrow \mathbf{Z}^p$  such that “ $\lim_{\Lambda_n \rightarrow \mathbf{Z}^p} \gamma_{\Lambda_n}(T, H)$ ” =  $\gamma$ . Actually to obtain a satisfactory answer one must introduce more general Gibbs ensembles which include “boundary conditions.” The convex closure  $K_{T,H}$  of the thermodynamic limits of such measures are called *Gibbs states* for our system at temperature  $T$  and

magnetic field  $H$ . In general these will correspond to the equilibrium states of our systems.

(B) Show that  $\lim_{\Lambda \rightarrow \mathbf{Z}^{\nu}} f_{\Lambda}(T, H)$  exists, independent of how  $\Lambda \rightarrow \mathbf{Z}^{\nu}$ , and describe its convexity and analytic properties. This is the free-energy per site  $f(T, H)$  for our system.

(C) What is the relation between  $K_{T,H}$  and the regularity properties of  $f$  at  $(T, H)$ ? For example, in our case  $K_{T,H}$  reduces to a point iff  $f$  is analytic at  $(T, H)$ . We say a phase transition occurs at  $(T, H)$  if  $f$  is not analytic at  $(T, H)$ .

One can classify the results applicable to general lattice models into roughly three groups. The first group contains general results which will apply to a large class of interactions for which the Lenz-Ising model would be a very special case. For example for a large class of lattice systems whose interactions are not too long range, one can define Gibbs states and a free-energy (called *pressure* with book under review) as above. Typical general results that one can prove are: (i) the Gibbs states form a Choquet simplex which implies that arbitrary Gibbs states have unique integral decompositions into extremal Gibbs states (= *pure phases*); (ii) Gibbs states are characterized by the DLR (= Dobrushin-Lanford-Ruelle) equations which shed considerable light on the structure of the measures; (iii) one can characterize  $\mathbf{Z}^{\nu}$ -invariant Gibbs states in terms of a variational principle involving the free-energy and entropy; (iv) various conditions involving correlations, etc. can be given which are equivalent to the Gibbs states reducing to a point.

The second group of results requires some special conditions on the interaction. The ferromagnetic condition  $J > 0$  on the Lenz-Ising model is a typical example. Examples of such results are the Peierls' argument proving the existence of phase transitions for a large class of ferromagnetic systems when  $H = 0$  and at sufficiently low temperature and the Lee-Yang theorem which proves the analyticity of  $f(T, H)$ ,  $H \neq 0$  for another large class of ferromagnetic systems. Thus for these systems phase transitions will not occur when  $H \neq 0$ . The third group of results are explicit calculation of various thermodynamic functions. This can only be done for a very limited number of models which does include the Lenz-Ising model with  $\nu = 1$  or 2. The case where  $\nu = 2$  is the famous Onsager calculation of the free energy at zero magnetic field.

We now turn to the book under review. The author is one of the leading researchers in mathematical statistical mechanics and is eminently qualified to write a definitive book on the foundations of the statistical mechanics of lattice systems. The book consists of seven chapters plus appendices. The first four chapters contain results which belong to the first group discussed above. Most of the general theorems one could hope for are proved for classical lattice systems on  $\mathbf{Z}^{\nu}$  provided their interactions are not "too long-ranged." Chapter 5 restricts itself to translation-invariant lattice systems on  $\mathbf{Z}$  with a slightly stronger decay condition placed on the interactions. The results in this chapter belong more to the second group of results rather than the first. In particular these results can be said to be definitive. The ergodic-theoretic properties of the equilibrium states with respect to the action of  $\mathbf{Z}$  on  $\Omega$  are completely described and the pressure is real analytic in the interaction

parameters. {The actual analyticity statement is more sophisticated since the interactions are parametrized by an infinite dimensional Banach space.} The proofs in Chapters 1–5 are elegant and complete but sometimes rather demanding of the reader.

In Chapter 6 the formalism of the first four chapters is partially extended to compact metrizable spaces with a  $Z'$ -action. An interesting wedding between the statistical mechanical formalism and topological dynamics is achieved. In Chapter 7 the richer formalism of Chapter 5 is extended to certain  $Z$ -actions on Smale spaces. Most detailed proofs in the last two chapters are omitted but complete references are given. Exercises, some of them quite difficult, are given at the end of each chapter. There are also complete bibliographical notes at the end of each chapter.

This is a beautiful but austere book. It is very much in the spirit of the Bourbaki treatise. We must compare this impression with the statement of the editor in the general preface to this Encyclopedia. It states: “Clarity of exposition, *accessibility to the nonspecialist* (italics added), and a thorough bibliography are required of each author.” If a person can learn a subject for the first time by reading Bourbaki, then perhaps that person can learn the statistical mechanics of lattice systems by reading this book. In this reviewer’s opinion most people will most profitably read Bourbaki and/or this book at the culmination of the learning process not at the beginning.

#### REFERENCES

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*Modular forms and functions*, by Robert A. Rankin, Cambridge University Press, Cambridge, London, New York, Melbourne, 1977, xiii + 384 pp., \$34.00.

*Modular functions and Dirichlet series in number theory*, by Tom M. Apostol, Graduate Texts in Mathematics, Number 41, Springer-Verlag, New York-Heidelberg-Berlin, 1976, x + 198 pp., \$14.80.

My first actual conversation with Mordell took place early in the 1960’s, when we were introduced (by L. C. Young, I believe) in the lounge of the Mathematics Research Center in Madison, Wisconsin. Always interested in the work of young mathematicians—a phrase that applied to me then—Mordell asked about my research interests. To my answer he responded with surprise (possibly feigned, it occurred to me later; in any event the point is the same), saying in effect—I don’t recall the exact words—“modular functions? I thought that was all settled years ago!”

That no mathematician, not even a Mordell in jest, could respond that way today is a measure of the extraordinary resurgence of interest that the field