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The metric theory of Banach manifolds, by Ethan Akin, Lecture Notes in Math., vol. 662, Springer-Verlag, Berlin and New York, 1978, xix + 306 pp., $13.50.

Few books have been written on the subject of an abstract model for the structure of manifolds of maps; the present book is one of this select group. Most of the books and papers in global analysis have been concerned with the point-set or differential topology of abstract Banach manifolds, with applications to various equations from physics, or else with the introduction and investigation of new examples of manifolds of maps which may play a role in future investigations. While this mass of literature is for the most part not concerned with the structure of manifolds of maps itself, it has nonetheless helped to shed some light on the nature of this still poorly-understood structure (and, equally important, on what this structure is not). Before discussing *The metric theory of Banach manifolds*, let me review what the literature of global analysis has told us thus far about the nature of manifolds of maps.

Let us assume for the moment that M_1 and M_2 are smooth finite-dimensional manifolds, and that M_1 is compact (possibly with boundary). Then all the standard examples of manifolds of maps from M_1 to M_2 contain the C^∞ maps, are infinite-dimensional manifolds, and are sandwiched as topological spaces between $C^\infty(M_1, M_2)$ and $C^0(M_1, M_2)$. By a result due to Palais [**14**, Theorem 16], it follows that all of these manifolds are of the same homotopy type as $C^0(M_1, M_2)$. But it is also well known to infinite-dimensional topologists [**9**] that any two homotopically equivalent topological manifolds, each of which is modeled on a separable infinite-dimensional Fréchet space (not necessarily the same space), are homeomorphic. While there exist important examples of manifolds of maps which are not separable, it still follows that most of the interesting spaces of maps from M_1 to M_2 are homeomorphic to $C^0(M_1, M_2)$. Thus nothing is gained topologically by investigating any Fréchet manifold of maps from M_1 to M_2 other than $C^0(M_1, M_2)$. Any gain is going to come from the analytical structure on the function space. Putting aside for the moment the consideration of global structures such as Finsler metrics (which can only be introduced once one already has a well-defined local structure), the basic analytical structure on a manifold of maps is local and is determined by the nature of the transition functions for the given atlas of coordinate charts on the function space. Thus the problem of deciphering the nature of the basic structure of manifolds of maps may be viewed in part as a problem in nonlinear functional analysis. Once this information is in hand, a global structure theory of manifolds of maps–infinite-dimensional differential geometry, really–will be within reach (for indications of how this might differ from a straightforward theory of Banach manifolds and Finsler metrics, see **[4]**, **[17]**, and **[19]**).

Incidentally, since the transition functions on manifolds of maps are induced by differential operators of order zero, and since a nonlinear differential operator of order k may be viewed as the composition of a linear differential operator of order k and a nonlinear differential operator of order zero, the study of these transition functions is very closely tied to the study of nonlinear differential operators (see [**15**, Chapter 15] for a readable development of this subject).

A property of the transition functions which seems to play a key role in the structure of manifolds of maps was uncovered by Palais ([**16**], [**15**, Lemma 19.12]): let ζ, η be smooth vector bundles over the compact manifold M, f: $\zeta \to \eta$ a smooth fiber-preserving map, and let $\mathfrak{M}(\cdot)$ denote any of the well-known section functors which assigns to each vector bundle over M a Banach space of continuous sections of the bundle. Then the induced map $\mathfrak{M}(f)$: $\mathfrak{M}(\zeta) \to \mathfrak{M}(\eta)$ sends bounded sets in $\mathfrak{M}(\zeta)$ to bounded sets in $\mathfrak{M}(\eta)$. Roughly speaking, this means that the transition functions map bounded sets to bounded sets.

Other key properties derive from the fact that these manifolds do not exist in isolation: there exist scales of manifolds sandwiched between $C^\infty(M_1, M_2)$ and $C^0(M_1, M_2)$, and the natural atlas on each manifold is obtained from the atlas on $C^0(M_1, M_2)$ by restriction of the charts in this atlas to the smaller space. So, for any manifold, the transition functions are continuous (and differentiable) not only for the given linear topological structure on the model space but for a great many weaker linear topological structures as well. Still further relations follow because the norm on a given model space is often derived in some fashion from the weaker norms (e.g. the C^k-norm from the C^0-norm, the L^p_k-norm from the L^p-norm, etc.).

This suggests two possibilities to the researcher interested in an application of this elusive structure to a problem in nonlinear analysis: the utilization of a Banach manifold of maps appropriate to the particular problem (such as a space of C^k or Sobolev maps) and careful investigation of additional properties imposed on the transition functions by conditions such as the ones just mentioned; or alternatively, the selection of the common core within all the manifolds of maps from M_1 to M_2–namely, $C^\infty(M_1, M_2)$– and investigation of the appropriate functional-analytic problem directly on this core. The first approach has been pursued in some detail by the reviewer **[6]** in connection with applications to the calculus of variations; the second approach has been taken by many authors–indeed, by anyone who has ever investigated differential calculus in Fréchet spaces, since the study of $C^\infty(M_1, M_2)$ reduces essentially to the study of the analytical structure of the transition functions on the model space. The most exciting work of the latter type has been done by R. Hamilton. In somewhat unrelated papers ([7], [8]) he develops two inverse function theorems in graded Fréchet spaces, one of which is tailored essentially for application to nonlinear elliptic operators, the other (more recent) one of which is a more general result of the Nash-Moser type (since Hamilton proved this theorem, several fairly complicated generalizations by other authors have appeared).

So far, however, neither of these approaches–selecting a single manifold of maps and concentrating on it alone–has proven adequate to deal with the problem of existence and uniqueness of flows for vector fields on function spaces which are induced by various nonlinear parabolic and hyperbolic differential operators of mathematical physics. While these vector fields are continuous and even differentiable on the space of C^∞ maps, there are no techniques currently available for dealing with vector fields on any sort of nonnormable Fréchet space. And while flows for these equations have been constructed by regarding the equations as unbounded vector fields on appropriate Banach function spaces (see, in particular, [10], [11], [18]), these techniques require the use of at least two distinct manifolds in the function space scale. A determination of the relation of these theorems to the inverse function theorems of [7], [8], or [6] might shed further light on the relation between the Banach manifolds in the various scales between $C^\infty(M_1, M_2)$ and $C^0(M_1, M_2)$, or else on the relation of the core space $C^\infty(M_1, M_2)$ to the rest of the function space manifolds. However, even the existence of any relation between the theorems on flows and the just-cited inverse function theorems is conjectural at this point.

In connection with the topic of flows for unbounded operators, it is appropriate to mention a result of J. Marsden [13], who has constructed a flow for the sum of a bounded and an unbounded operator from the flows for the individual operators by a Lie product technique, something like P. Chernoff's nonlinear generalization of the Trotter product formula. However, the theorems of Chernoff and Trotter are very different in that they show the convergence of a Lie product approximation scheme to a flow which is already assumed to exist. Marsden's technique requires the use of at least four distinct manifolds in a scale; information about relations between the spaces may be present implicitly in the hypotheses of the various lemmas and theorems of his paper.

A major difficulty in the development of the appropriate analytical tools for global analysis is the fact that many theorems which are inequivalent in infinite-dimensions reduce to the same result in finite-dimensional spaces, and as a consequence it is often difficult to decide what should be the appropriate generalization to infinite-dimensional spaces of a given finite-dimensional result. For example, the theorems on flows for unbounded operators cited above, and the usual existence and uniqueness theorem for flows generated by locally Lipschitz vector fields on Banach spaces, both reduce to the same result for vector fields on finite-dimensional spaces, though in infinite dimensions the theorems for unbounded vector fields are much more general (modulo the fact that the theorems of [10] and [11] apply only to reflexive spaces). As another example, compare the inverse function theorem of [6] with the usual inverse function theorem for C^k mappings between Banach spaces: again, the two theorems reduce to the same result in finite dimensions, though for infinite-dimensional spaces the second result is more general but with a weaker conclusion than the first.

In addition to the sort of investigation just discussed, there has been work done in extending the construction of a manifold structure on various spaces of maps from M_1 to M_2 to the cases where M_1 and M_2 are allowed to be noncompact, or even infinite-dimensional. The case of M_1 compact, M_2 arbitrary, was first treated by H. Eliasson [5]. His construction of the manifold structure on spaces of maps from M_1 to M_2 roughly parallels the development by Palais in *Foundations of global non-linear analysis* for the case of M_2 finite-dimensional, the main difference being that Eliasson had to postulate the existence of a smooth spray on M_2, the existence of which is not automatic as in the finite-dimensional case. An ingenious alternative method of construction which avoided the use of an exponential map on M_2, and hence also of a spray, was developed by N. Krikorian [12].

The structure of spaces of mappings from M_1 to M_2 becomes much more delicate when M_1 is permitted to be noncompact. In this case it is necessary to impose a metric on M_2 just to obtain a topological manifold structure on $C^0(M_1, M_2)$. For this manifold, a necessary condition for two maps $f, g \in C^0(M_1, M_2)$ to be in the same component of the function space is that $\sup \rho(f(x), g(x)) < \infty$, where ρ is the metric on M_2. For the case of M_1 and M_2 noncompact, it is always possible to change the metric on M_2 in such a way as to change the components of $C^0(M_1, M_2)$. And once one progresses to sets of differentiable maps from M_1 to M_2 (C^k maps, Hölder maps, etc.), it is necessary to have at least a metric on M_1 as well if there is to be any hope of equipping a given set of such maps with a manifold structure.

As far as I am aware, only two researchers, M. Cantor and E. Akin, have undertaken the investigation of manifolds of maps on a noncompact domain. Their techniques are totally unrelated, as are the types of function spaces to which their ideas apply. Cantor has been interested in studying various parabolic and hyperbolic differential equations of mathematical physics (See [1], [2], and especially [3]) defined on finite-dimensional noncompact domains (usually assumed to be diffeomorphic to $\mathbf{R}^n$, though not necessarily equivalent to $\mathbf{R}^n$ as Riemannian manifolds). For these applications, he has found it convenient to work with manifolds modeled on what he refers to as "weighted Sobolev spaces". These model spaces admit norms which are defined by integrals over M_1, are closely related to the usual Sobolev spaces, and are reflexive (subject to the familiar restriction $p \in (1, \infty)$). For each of these model spaces, the fact that the norm on the space is defined by an integral over M_1 (M_1 is assumed to be a complete Riemannian manifold, the measure on M_1 the one induced by the Riemannian metric, so the measure of the whole manifold is infinite) guarantees that $f(x) \to 0$ as $x \to \infty$. It follows that the manifolds of maps which Cantor constructs are made of maps which exhibit the same asymptotic behavior as $x \to \infty$ in M_1. This type of manifold can only be constructed when M_1 is finite-dimensional; indeed, Sobolev-type spaces of continuous functions have never been satisfactorily generalized to infinite-dimensional domains.

E. Akin, on the other hand, introduces us in *The metric theory of Banach manifolds* to the study of manifolds of maps from M_1 to M_2 with no restrictions whatsoever upon the dimensionality of M_1 and M_2. He has taken an axiomatic approach to the construction of the manifold structure much like the approach Palais pioneered in [15], although Akin's axioms are necessarily more intricate because of the possible infinite-dimensionality of M_1 and M_2. Thus Akin considers an abstract class of mappings $\mathfrak{M}(M_1, M_2)$ rather than any specific type of map; however, the reader should keep in mind that these axioms have been tailored to fit the examples of C^k, Lip_k, and Hölder maps (which, unlike Sobolev maps, make sense on infinite-dimensional domains). The norms on the model spaces for these examples are defined locally in terms of behavior on M_1, so it is not surprising that Akin does not assume any asymptotic behavior on the mappings in $\mathfrak{M}(M_1, M_2)$ near infinity (because of his interest in applications to foliations, he would probably find any constraints which would impose asymptotic behavior totally unacceptable). This treatment is intended to apply only to those classes of mappings which are closed under composition, which again excludes mappings of the Sobolev type even in the case of M_1 finite-dimensional.

The construction of the atlas on $\mathfrak{M}(M_1, M_2)$ proceeds under the assumption that each manifold M_i possesses an atlas $\mathcal{Q}_i$ (smaller than the maximal atlas compatible with the given differentiable structure on M_i) such that there is a uniform bound on the norm of each transition function and a sufficiently large number of its derivatives (the exact numerical value of "sufficiently" depending upon the section functor $\mathfrak{M}$), together with restrictions on the size of the domains of the transition functions which are specified in terms of the metric structures on M_1 and M_2. Technical details make the construction extremely complicated; it is to be hoped that someone will be able to capture the essence of this nontrivial theory in a more easily accessible development (discussions of the more subtle aspects of some of the definitions plus inclusion of more examples would be helpful). For instance, for the case where M_1 and M_2 are assumed to be finite-dimensional, it might be possible to adapt Cantor's techniques to Akin's setting to obtain a significant simplification of method. Alternatively, in the direction of greater generality, it might be possible to adapt Krikorian's technique [12] to this infinite-dimensional setting to remove the dependence of Akin's construction upon the existence of a well-behaved exponential map on the space M_2.

Chapter VII of *The metric theory of Banach manifolds* is devoted to the development of the technical machinery necessary to deal with applications to foliation theory. The author promises to develop relations between his metric structures and the geometrical structures of his manifolds of maps in future papers.

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Permanents, by Henryk Minc, Encyclopedia of Mathematics and its Applications (Gian-Carlo Rota, Editor), Volume 6, Addison-Wesley, Reading, Mass., 1978, xviii + 205 pp., $21.50.

The year 1979 can be regarded as the 20th anniversary of the *theory* of the permanent function. True, permanents were introduced in 1812 by Binet [**2**] and Cauchy [**9**], and several identities, usually involving determinants as well, were obtained in the 19th century by some ten other mathematicians including Cayley and Muir. Indeed it was Sir Thomas Muir [**30**] who in 1882 coined the term 'permanent' for the following function defined on $n \times n$ matrices $A = [a_{ij}]$:

$$\operatorname{per} A = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

where the summation extends over all $n!$ permutations σ of $\{1, \ldots, n\}$. True, in 1903 Muirhead [31] obtained the following beautiful result. Let $c = (c_1, \ldots, c_n)$ be a positive n-tuple, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta =$