The book contains no exercises, partly because the routine and straightforward computations that are usually left to the reader are worked out in the text, and partly because it is not the intention of the author to refer to results in the theory of Lie algebras not covered in the text.

Unfortunately, there is a large number of misprints in the text and many symbols have been left out. Apart from some wrong definitions, the table containing the Coxeter-Dynkin diagrams for the complex simple algebras gives the wrong diagram for E_8. In the bibliography the name of J. Dixmier is misspelled and some French words have wrong accents.

The efforts of the author to bring down an important subject to the level of readers without a general background in Mathematics are commendable, but any reader of this book may get lost in its forest of mathematical equations.

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Homology and cohomology theory, by William S. Massey, Monographs and Textbooks in Pure and Applied Mathematics, no. 46, Marcel Dekker, Inc., New York, 1978, xiv + 412 pp. $29.75.

Algebraic topology attempts to solve topological problems using algebra. To do so requires some sort of machine which produces the "algebraic image" of topology, and it is the machine itself on which topologists often spend most of their time, first carefully building and then diligently refining. Historically, the first such machine was ordinary homology and cohomology theory. (The word "ordinary" is not a slur–it means homology and cohomology defined from an algebraic chain complex as opposed to "extraordinary" theories such as K-theory.) Yet in the 84 years since homology was first mentioned, algebraic topology has developed rapidly and diversely. In the past 30 years this development has been particularly apparent with the problems more diverse and the machinery more and more complex, going far beyond its humble origins. Indeed, at the moment the subject is a tinkerer's delight; one can choose a machine and modify it almost at will.

Precisely when the machine works; or how it is related to other parts of the subject, is often not quite known. The phrase "nice space" seems to be used with increasing frequency in algebraic topology. What is the most efficient way to develop the machinery? What is the best way to teach it to graduate students or to explain it to other mathematicians? All this is often forgotten in the frenzy to answer the next question. Exposition and careful development of the foundations have often appeared in unpublished lecture notes, and copies turn into prized possessions. Sad to say, there is no glory in cleaning up after a party.

Homology and cohomology theory is a cleansing performed at the very roots of algebraic topology. It develops ordinary homology and cohomology theory in a neat and orderly fashion from the beginning, and does so in a novel way which is technically very pleasant. But why should such an old and established area of topology require cleansing at all? The answer lies in the

somewhat complicated development of homology theory which took place some 40 years ago.

The notion of homology was first introduced by Poincaré (1895) in his now famous paper (together with 5 sequels.) He considered only polyhedra, spaces which are given as a collection of simplices in some Euclidean space, and the algebraic chain complex from which the homology groups were derived was virtually presented to one along with the space itself. This was, of course, simplicial homology. Technically Poincaré never actually mentioned homology groups–he worked with the Betti and torsion numbers which determine the groups. Regardless of the language he used, the idea was clearly presented. It was an idea which would be refined and generalized during the next 50 years into a profusion of theories which were individually enormously useful, but as a whole thoroughly confusing.

At the outset the applications of homology theory were to manifolds, polyhedra which are locally Euclidean. The key to such applications lay in duality. In his original paper Poincaré formulated the fundamental duality theorem: for a closed, connected, two-sided (= orientable) n-manifold M^n the Betti numbers $\beta_k(M^n)$ and $\beta_{n-k}(M^n)$ are equal, for each k, and similarly the torsion numbers satisfy a duality relation, but between dimensions k and $n - k - 1$. It does not take long for a student of algebraic topology to realize how valuable Poincaré Duality is. (For example, a closed, connected, two-sided 3-manifold which is simply connected must have the same homology as the 3-sphere.) Some time later Alexander (1922) proved another remarkable duality theorem: for any subpolyhedron K of the n-sphere S^n, $\beta_k(K) = \beta_{n-k-1}(S^n - K)$ for each k and a similar duality holds for the torsion numbers with a shift in dimension. Finally Lefschetz (1930) combined these two duality theorems into one, and in a sequence of papers formulated and proved yet another duality result, Lefschetz Duality. In many ways duality was not only at the heart of the applications but was also the prime motivation for variation of the fundamental theory.

First there was the algebraic innovation of homology with coefficients. The idea of Tietze (1908) and Alexander and Veblen (1913) was very simple: simply reduce the chain complex modulo 2 and *then* compute the "Betti" numbers. What on earth for? (That's a question many beginning graduate students ask on first encounter with coefficients.) They both explained why. The usual Poincaré Duality does *not* hold for one-sided (= nonorientable) manifolds, but with Z_2 coefficients it works quite well. It was a short step to the introduction of Z_n coefficients by Alexander (1926) and finally of coefficients in any abelian group by Čech (1935). The latter innovations provided, among other things, a natural systematic way to deal with the mysterious torsion numbers.

Duality also brought forth a companion for homology. The idea of cohomology, to consider the dual of the chain complex and compute *its* homology groups, originated with the "pseudocycles" of Lefschetz (1930). It was developed by Alexander (1935) and Whitney (1938), who was the first to use the word "cohomology". (It is Whitney, therefore, who has the dubious distinction of starting a rather unpleasant trend for mathematicians to add the prefix "co-" to any and all mathematical terms.) For duality cohomology

offered a supreme simplification and unification. Poincaré Duality now becomes simply $H^k(M^n) \approx H_{n-k}(M^n)$, and Alexander Duality becomes $H^k(K) \approx H_{n-k-1}(S^n - K)$.

In a certain sense all these innovations were designed merely to refine Poincaré's original machine. To a large extent the generalizations were dictated in a natural and consistent way by duality. Things were not so simple, however, when one tried to extend not the machine itself, but rather the spaces to which it could be applied.

It was of course natural that one should try to define homology and cohomology for arbitrary topological spaces. The most widely known "expansion" is singular theory. The basic ideas of singular theory are contained in Veblen (1922), but the theory was really developed first by Lefschetz (1933) and then by Eilenberg (1944). Briefly, the idea is to associate to a space a (somewhat artificial) giant complex built from all continuous maps of simplices into the space.

There was a second, competing expansion of homology and cohomology; first by Vietoris (1927), Alexandroff (1928) and Pontrjagin (1931) to compact metric spaces; and then by Čech (1932) to arbitrary spaces. Once again the idea was to associate a giant complex to any space, but this time one built a simplicial complex by considering all open covers of the space. There was in fact another quite different way to define Čech cohomology which was begun by Alexander (1935) and developed by Spanier (1948). In this approach one built a complex by considering mappings of finite cartesian products of a space with itself into some abelian group. The difficulty with this approach was that it was not apparent how to develop homology theory in a similar way.

These then were the two main variants for expansions–singular theory and Čech theory–each with its own minor variations. Naturally all variants agreed with Poincaré's homology on polyhedra but, sad to say, they did *not* agree in general. There were spaces for which the homology groups of the two theories were decidedly different. Which theory was to be preferred? The answer is not so simple.

From many points of view one might argue for Čech theory. In particular, since duality dictated the previous refinements of Poincaré homology, why not let it decide the issue here? Alexander Duality should certainly generalize to: for any compact subset K of S^n, $H^k(K) \approx H_{n-k-1}(S^n - K)$. This *is* true ... but *only* if one uses Čech cohomology! For this reason (as well as many others) Čech theory has many advantages over singular theory.

On the other hand, the argument is not entirely one-sided. Čech theory is an extraordinarily cumbersome machine to handle and is certainly not intuitive. (This is really an argument concerning pedagogy). There is, however, an even more important defect. Čech homology theory, as defined by Čech, does not satisfy the exactness axiom for homology unless one severely restricts the coefficient group G. (See Eilenberg and Steenrod (1952), Chapter X for a complete discussion.) It is a major shortcoming which is not shared by singular theory.

We find ourselves at the very roots of algebraic topology faced with a dilemma. Which is the best expansion of Poincaré's homology theory, both

technically and pedagogically? Massey gives us his answer, laid out neatly and developed with care in this book. Fortunately one doesn't have to read all 410 pages to glimpse his solution. In fact, one doesn't have to read the book at all! In an excellent article (Massey (1978)) the author has not only indicated why his answer is a good one but also precisely how some recent work of Nöbeling has allowed the formulation of the correct variant. (The answer, by the way, is to use the Alexander-Spanier definition of Čech cohomology, suitably modified so that the resulting homology theory *does* satisfy the exactness axiom. It is Nöbeling's work which enables one to perform this miracle.) It should be added, perhaps, that while the article serves as an excellent preface to the book, one would certainly need to read the book itself for a thorough development of the theory, first to locally compact spaces and then to general spaces.

Now let me add some perspective to all this. I have stated that homology and cohomology are at the very foundation of algebraic topology, and, moreover, that topologists have been confused for some time about which particular variant to use. Some readers are apt to conclude that, (i) algebraic topology is a subject in great disarray, (ii) algebraic topologists have certainly been a careless and giddy lot to allow such a situation to continue, and (iii) the news of this breakthrough will surely excite all topologists. Lest the reader be led astray it should be forcefully pointed out that *most* of the time *most* topologists only consider spaces for which all theories *do* agree. Many machines of algebraic topology breakdown when applied to general spaces; the solution is to restrict one's attention, whenever possible, to a more limited collection of spaces–all else is pathology. For most topologists, (most of the time), there simply *is* no dilemma! It will probably come as no surprise, therefore, that many algebraic topologists will not find this definitive treatment of homology and cohomology immediately valuable in their current work, and, moreover, will view it as a very elegant, yet rather technical improvement on an old subject.

What is the value of such work? First, there are times when, no matter how nice the spaces one begins with are, the spaces one ends with *might* be pathological. (For example, fixed point sets of homeomorphisms.) When one needs the full force of ordinary homology and cohomology it's nice to have the proper machine set-up and waiting, not as inaccessible lecture notes but in a book. But perhaps equally important is the principle involved–the principle that at some point in the development of any mathematical area one must go back to tidy up simply because it's mathematically sound to do so. Massey has done this at the very roots of algebraic topology and the results are techincally very satisfactory and pedagogically reasonable. Yet, as we indicated above, algebraic topology has developed far beyond its roots and in the process much debris has been left behind. There is a great need for simplification, unification and illuminating exposition; it is a need which is largely unmet. Perhaps it's time for some to stop partying just long enough to tidy up.

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The collected papers of Alfred Young 1873–1940, G. de B. Robinson (editor), University of Toronto Press, Toronto and Buffalo, 1977, xxvii + 684 pp., $10.00.

The twenty seven papers of the Reverend Alfred Young are attractively collected in this volume together with a foreword by G. de B. Robinson and Young's obituary by H. W. Turnbull. The papers were all written over forty years ago (although one was published posthumously in 1952), and as Turnbull says in the obituary:

"Young's work is never easy reading, for it lacks that quality which helps the reader grasp the essential point at the right time. The very closest and constant attention is required to pick out some of the most fundamental results from a mass of detail. One could almost suppose that he camouflaged his principal theorems. His work resembles a noonday picture of a magnificent sunlit mountain scene rather than the same in high relief with all the light and shade of early morning or sunset."

It is natural then to ask whether it is worthwhile to publish a volume of old obscure papers. To answer this we shall examine some of the ramifications of Young's ideas in recent research. First of all two recent conferences *Combinatoire et representation du groupe symetrique* in Strasbourg [19] and *Alfred Young Day* in Waterloo [82] were both centered on the theme of Young's research.