RESEARCH ANNOUNCEMENTS

ERGODIC THEORY OF AMENABLE GROUP ACTIONS. I:
THE ROHLIN LEMMA

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Classically, ergodic theory began with the study of flows or actions of $\mathbf{R}$. Later, for technical reasons, much of the theory was first developed for actions of $\mathbf{Z}$. More recently, there has been interest in extending the theory to actions of more general groups such as $\mathbf{Z}^d$, $\mathbf{R}^d$, abelian groups etc. (see e.g. [G], [K], [KW], [L]). The natural setting seems to be amenable groups and we have been able to do much of the theory in that generality. We report here on the discrete case, so henceforth G will always be countable, and focus on, what has turned out to be a very basic tool in the modern developments, the Rohlin lemma. Let G denote a countable amenable group, $(X, \mathcal{B}, \mu)$ a finite nonatomic Lebesgue measure space, and $T\colon G \times X \longrightarrow X$ a measure-preserving action of G on X. We will usually suppress T and write simply gx for $T(g, x)$. Recall that the action is *free* if for μ a.e. x, $gx \neq x$ for $g \neq 1 \in G$. Rohlin's lemma is valid for a finite set $F \subset G$, and a free action of G on $(X, \mathcal{B}, \mu)$ if for all $\epsilon > 0$ there is a set $B \in \mathcal{B}$ such that

(i) $\{fB\colon f \in F\}$ are disjoint,

(ii) $\mu(FB) \equiv \mu(\bigcup_{f \in F} fB) > 1 - \epsilon$.

We say that F *tiles* G, or that F is a *tiling set* if there is a set of centers $C \subset G$ such that $\{Fc\colon c \in C\}$ is a partition of G.

PROPOSITION 1. *If Rohlin's lemma is valid for a finite set F for some free action of G then F tiles G.*

THEOREM 2. *If F tiles G, and G is amenable, then Rohlin's lemma is valid for F and any nonsingular free action of G.*

Thus for amenable groups, Rohlin's lemma is entirely equivalent to a purely algebraic property-tiling. In applications one wants to apply Rohlin's lemma to sets that are almost invariant in the following sense: for $K \subset G$ and $\delta > 0$, F

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is said to be $(K - \delta)$-*invariant* if

$$\frac{|\{f \in F: \; Kf \subset F\}|}{|F|} > 1 - \delta.$$

PROPOSITION 3. *If G is a solvable group, or a finite extension of a solvable group, or an increasing union of such groups, then for any finite K and $\delta > 0$ there is a $(K - \delta)$-invariant tile.*

We were not able to prove Proposition 3 for arbitrary amenable groups. This is of course purely an algebraic question and may be not too difficult to settle (for algebraists). Since we were determined to carry out our program we came up with a somewhat weaker version of Rohlin's lemma which appears to be adequate. First, some more ϵ-definitions. Finite sets $\{A_1, \ldots, A_k\}$ will be said to be ϵ-*disjoint* if there are subsets $\overline{A}_i \subset A_i$ such that

$$|\overline{A}_i| > (1 - \epsilon)|A_i|, \quad 1 \le i \le k,$$

with the $\overline{A}_i$'s pairwise disjoint. This is implied by,

$$\left| A_{i+1} \cap \left(\bigcup_{j \le i} A_j \right) \right| \bigg/ |A_{i+1}| \le \epsilon, \quad 1 \le i < k.$$

A collection $\{B_1, \ldots, B_l\}$ is said to α-*cover* C if

$$\frac{\left| C \cap \left(\bigcup B_i \right) \right|}{|C|} \ge \alpha.$$

A collection $\{T_1, \ldots, T_M\}$ will be said to ϵ-*tile* the group G if any finite set in G can be $(1 - \epsilon)$-covered by an ϵ-disjoint collection of right translates $\{T_i c_{ij}\}$ of the T_i.

PROPOSITION 4. *If G is an amenable group and $\epsilon > 0$ is given, then there exists an N, depending only on ϵ, such that for all finite K and $\delta > 0$, there exist $\{T_1, T_2, \ldots, T_N\}$ that ϵ-tile G with each T_i being $(K - \delta)$-invariant.*

In fact one can be a bit more precise, to wit: it can be arranged so that the translates of T_i are actually disjoint from the translates of T_j for $i \ne j$ with the only nontrivial intersections coming from within the translates of each T_i. Now with this more stringent condition we can prove

THEOREM 5. *If $\{T_1, \ldots, T_N\}$ ϵ-tiles G as above, and G is amenable then for any free, measure-preserving action of G on $(X, \mathcal{B}, \mu)$ there are sets $B_1, \ldots, B_N$ such that*

(i) $\{T_i B_i: \; 1 \le i \le N\}$ *are disjoint,*

(ii) $\{t B_i: \; t \in T_i\}$ *are ϵ-disjoint, $1 \le i \le N$,*

(iii) $\mu(\bigcup T_i B_i) > 1 - 10\epsilon$.

If the action is merely nonsingular then to cover $1 - \epsilon$ of the space, as in (iii), the collection of T_i's has to ϵ^2-tile the group. It is helpful to think of (ii) in the following way: each set B_i can be broken up into pieces, B_{ij}, $1 \leqslant j \leqslant J$, with a $T_{ij} \subset T_i$ corresponding to each piece such that

(i) the $T_{ij}B_{ij}$, $1 \leqslant j \leqslant J$, are disjoint,
(ii) $\{tB_{ij}: \ t \in T_{ij}\}$ are disjoint,
(iii) $|T_{ij}|/|T_i| > 1 - \epsilon$.

Thus, one has lots of different Rohlin towers, which is still good enough for the applications we have in mind for the entropy theory and the isomorphism theory of Bernoulli actions. What is important is that there is control on the number of different towers—namely it is exponentially small in the number of elements of T_i.

There is a stronger version of the Rohlin lemma, that was first proved for $\mathbf{Z}^d$ actions in [KW], that has proved to be very useful. It says that given a finite partition P of the space X, one can construct the Rohlin tower in such a way that each level is independent of P. There is such a stronger version both of Theorem 2 and of Theorem 5. We write out Theorem 2'; Theorem 5' is obtained in a similar way.

THEOREM 2'. *If F tiles G, and G is amenable, then for any free action of G on $(X, \ \mathcal{B}, \ \mu)$, any finite partition P of X, and any $\epsilon > 0$ there is a set $B \in \mathcal{B}$ such that*

(i) *$\{fB: \ f \in B\}$ are disjoint,*
(ii) *for each $f \in F$, and each $P \in P$, $\mu(fB \cap P) = \mu(fB) \cdot \mu(P)$,*
(iii) *$\mu(FB) > 1 - \epsilon$.*

As an application of these results, and some further refinements (to handle the case of nonfree actions), we have the following extension of Dye's theorem.

THEOREM 6. *Any nonsingular action of an amenable group is hyperfinite, i.e., orbit equivalent to a Z-action.*

Detailed proofs of these results, and their applications to a theory of entropy, and an isomorphism theorem for Bernoulli actions will appear in future publications; we give here some idea of how the proofs go. For Proposition 1, one looks at a typical orbit and asks when it hits the set FB, and then considers the consequences of F not tiling G.

At the heart of most of the other proofs lies the following elementary lemma.

LEMMA. *If the sets $E_1, \ldots, E_n$ form an even covering of X in the sense that $\Sigma_1^n \ 1_{E_i}(x)$ is a constant then for any $E \subset X$ there is some i for which $\mu(E_i \cap E)/\mu(E_i) \leqslant \mu(E)$. Moreover, for any $\epsilon > 0$, there is an ϵ-disjoint subcollection of the E_i's that ϵ-covers X.*

The lemma is what enables one to go from the overlapping towers that the freeness of the action guarantees to nearly disjoint towers. It is the amenability that allows one to iterate the ϵ-covering to get $(1 - \epsilon)$ of the space.

A last remark concerns Theorem 6. Following a presentation of some of these results at a recent conference held in Santa Barbara, in collaboration with A. Connes and J. Feldman, Theorem 6 has been extended to prove that any amenable equivalence relation is hyperfinite. That, together with applications to the theory of von Neumann algebras, will appear in a separate publication.

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