A TOPOLOGICAL CHARACTERIZATION OF REAL ALGEBRAIC VARIETIES

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We show that if a smooth locally conelike stratified set admits a certain kind of topological resolution then it is homeomorphic to a real algebraic set, i.e. zeros of polynomial functions (this generalizes [AK₁], [AK₂]). We expect that the algebraic resolution of singularities [H] implies that every algebraic set admits such a topological resolution, hence it is reasonable to suspect that we have a complete topological characterization of real algebraic sets.

Examples of some stratified sets admitting such a topological resolution are spaces which we call $A_k$-spaces, $k = 0, 1, 2, \ldots$. We define $A_k$-spaces inductively by saying that $A₀$-spaces are smooth compact manifolds, and an $A_k$-space is a compact smooth stratified set $X$ with a trivialization of a neighborhood of each stratum $X_\ell$, $h_\ell: X_\ell \times \text{cone}(\Sigma_\ell) \to X$ where $\Sigma_\ell$ is an $A_{k-1}$-space which bounds a compact $A_{k-1}$-space with boundary ($h_\ell$ required to be compatible with the trivializations of neighborhoods of the strata of $\Sigma_\ell$).

The topological resolution of an $A_k$-space $X$ is obtained by a sequence of 'blow ups' as follows: take a lowest dimensional stratum $X_\ell$ (the 'center' of the 'blow ups') with trivialization $h_\ell: X_\ell \times \text{cone}(\Sigma_\ell) \to X$ and replace $h_\ell(X_\ell \times \text{cone}(\Sigma_\ell))$ by $X_\ell \times W_\ell$ where $W_\ell$ is a compact $A_{k-1}$-space which $\Sigma_\ell$ bounds.

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bounds. We have a map from the new space to $X$ which is the identity outside image $(h_i)$ and which collapses $X_i \times S_i$ to $X_i \times \ast$ where $S_i$ is a spine of $W_i$ and $\ast$ is the vertex of $\text{cone}(\Sigma_i)$. After a finite number of such blow ups we obtain a smooth manifold $\tilde{X}$ and a ‘resolution’ $\tilde{X} = Z_n \rightarrow Z_{n-1} \rightarrow \cdots \rightarrow Z_0 = X$. We say $X$ is an $A$-space if it is an $A_k$-space for some $k$. In particular we prove:

**Theorem.** The interior of any compact $A$-space is homeomorphic to a real algebraic set. Furthermore the natural stratification on this algebraic set coincides with the stratification of the $A$-structure.

One of the reasons $A$-spaces are of interest is that Akbulut and Taylor have shown that any compact P.L. manifold has the structure of an $A$-space [AT]

**Corollary.** The interior of any compact P.L. manifold is P.L. homeomorphic to a real algebraic set.

**Sketch of Proof.** Take a resolution of a compact $A_k$-space $X$: $Z_n \rightarrow Z_{n-1} \rightarrow \cdots \rightarrow Z_0 = X$ where $Z_n$ is a smooth compact manifold. Each $Z_{i+1} \rightarrow Z_i$ has a certain ‘center’ a smooth manifold $X_i \subset Z_i$ along which a topological ‘blow up’ occurs. We construct a tower of nonsingular varieties $V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_0 = \mathbb{R}^n$ with $X_i \subset V_i$ as a nonsingular subvariety and imbeddings $Z_i \subset V_i$ which commute with projections (i.e. all the above maps given by arrows) and are in some sense ‘stable’ over the projections.

This means that if $Z_n$ is moved by a small isotopy in $V_n$ the image of $Z_n$ under the composite projection $\pi: V_n \rightarrow V_0$ is isotopic to $X$. We then approximate the submanifold $Z_n \subset V_n$ by an algebraic set $Q$; and then ‘blow down’ $Q$ algebraically to an algebraic set $V$ which is homeomorphic to $\pi(Q)$, which is in turn homeomorphic to $X$.

Each $V_{i+1}$ and the projection $V_{i+1} \xrightarrow{\pi} V_i$ is obtained by a certain algebraic ‘multiblowing-up’ process from $V_i$ along $X_i$. $X_i$ is a lowest dimensional stratum of $Z_i$. Given $Z_i \subset V_i$ the imbedding $Z_{i+1} \subset V_{i+1}$ is obtained roughly as follows: Let $h_i: X_i \times \text{cone}(\Sigma_i) \rightarrow Z_i$ be the neighborhood trivialization of $X_i$, and $W_i$ a compact $A_k$-space which $\Sigma_i$ bounds; imbed $X_i \times W_i$ into $V_{i+1}$ so that
(a) $X_i \times W_i$ is transverse to $\pi_i^{-1}(X_i)$,

(b) $\pi_i^{-1}(X_i) \cap (X_i \times W_i) = X_i \times (a \text{ spine of } W_i)$,

(c) $\pi_i(X_i \times W_i) \approx h_i(X_i \times \text{cone}(\Sigma_i))$.

Then extend this imbedding to an imbedding of $Z_{i+1}$ into $V_{i+1}$ by simply lifting the imbedding $Z_i$-image($h_i$) (° $Z_{i+1} - X_i \times W_i$) to $V_{i+1}$ via $\pi_i$ so that $\pi_i(Z_{i+1}) \approx Z_i$. In particular, along the way we prove that the $A_k$-space $\Sigma_i$, which bounds, necessarily has to bound an $A_k$-space $W_i$, which has a spine consisting of transversally intersecting codimension one $A_k$-subspaces without boundaries. Choosing such $W_i$'s enables us to show (a), (b), (c). Details are long and geometric in nature, they will appear in [AK3]. The proof applies to spaces more general than $A$-spaces, which leads us to believe that a satisfactory topological classification theorem for real algebraic sets is within reach.

REFERENCES


