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ON IRREDUCIBLE MAPS

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The notion of irreducible map was introduced by M. Auslander and I. Reiten in [3] and plays an important role in the representation theory of artin algebras.

We recall that an artin ring Λ is said to be an artin algebra if its center C is an artin ring and Λ is finitely generated left Λ-module. Now choose a complete set $P_1, \ldots, P_s$ of representatives of the isomorphism classes of indecomposable projectives in $\mathrm{mod}(\Lambda)$, we will denote by $\mathrm{pr}\ \Lambda$ the full subcategory of $\mathrm{mod}\ \Lambda$ whose objects are $P_1, \ldots, P_s$. A map $g: X \longrightarrow Y$ in $\mathrm{mod}(\Lambda)$ is said to be irreducible if g is neither a split monomorphism nor a split epimorphism and for any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & Y \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle h} & \\ & Z & \end{array}$$

f is a splittable monomorphism or h is a splittable epimorphism.

We study irreducible maps in $\mathrm{mod}(\Lambda)$ by using properties of the Jacobson radical of $\mathrm{mod}(\Lambda)$. We recall that the Jacobson radical of $\mathrm{mod}(\Lambda)$ is the subfunctor rad of the two variable functor $\mathrm{Hom}: (\mathrm{mod}(\Lambda))^{\mathrm{op}} \times \mathrm{mod}(\Lambda) \longrightarrow \mathrm{Ab}$ defined by

$$\begin{aligned} \mathrm{rad}(X, Y) &= \{f \in \mathrm{Hom}(X, Y) \mid 1 - gf \text{ is invertible for every } g \in \mathrm{Hom}(Y, X)\} \\ &= \{f \in \mathrm{Hom}(X, Y) \mid 1 - fh \text{ is invertible for any } h \in \mathrm{Hom}(X, Y)\}. \end{aligned}$$

It is easy to prove that if X and Y are indecomposables, then $\mathrm{rad}(X, Y)$ consists of all nonisomorphisms, from X to Y.

We can prove the following result:

PROPOSITION 1. *Let C and D be indecomposables in* $\mathrm{mod}(\Lambda)$. *Then*

(a) *A map $f: C \longrightarrow$ is irreducible iff $f \in \mathrm{rad}(C, D)$ and $f \notin \mathrm{rad}^2(C, D)$, where $\mathrm{rad}^2(C, D)$ consists of all maps of the form $t_1 t_2$ with $t_2 \in \mathrm{rad}(C, X)$ and $t_1 \in \mathrm{rad}(X, D)$.*

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(b) *A map*

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} : C \longrightarrow D \amalg \cdots \amalg D$$

is irreducible iff $\bar{g}_1, \ldots, \bar{g}_n$, $\mathrm{rad}(C, D)/\mathrm{rad}^2(C, D)$ *are linearly independent over* $\mathrm{End}(D)/\mathrm{rad}\ \mathrm{End}(D)$.

Using properties of rad we obtain the result stated below:

THEOREM 1. *Suppose* $f \in \mathrm{rad}(X, Y)$. *Then the following statements are equivalent*

(a) *f is irreducible,*

(b) *For any splittable monomorphism u: $C \longrightarrow X$ with C indecomposable the composed map fu is irreducible.*

(c) *For any splittable epimorphism v: $Y \longrightarrow D$ with D indecomposable vf is irreducible.*

If the artin algebra Λ is infinite and of finite representation type then we have rather precise information about irreducible maps between indecomposables in $\mathrm{mod}(\Lambda)$. We are able to prove the following

THEOREM 2. *Suppose Λ is an infinite artin algebra of finite representation type and let X and Y be indecomposables in* $\mathrm{mod}(\Lambda)$. *If we denote by d and d' the dimensions of* $\mathrm{Hom}(X, Y)/\mathrm{rad}^2(X, Y)$ *over* $\mathrm{End}(X)/\mathrm{rad}\ \mathrm{End}(X)$ *and over* $\mathrm{End}\ (Y)/\mathrm{rad}\ \mathrm{End}(Y)$ *respectively, then* $dd' \leqslant 3$.

We also get information about the middle term of any almost split sequence in $\mathrm{mod}(\Lambda)$. We recall that the short exact sequence $0 \longrightarrow A \xrightarrow{u} B \xrightarrow{V} C \longrightarrow 0$ is said to be almost split if (a) the sequence does not split. (b) For any h: $X \longrightarrow C$ nonsplittable epi there exists g with $vg = h$. (c) For any h': $A \longrightarrow Y$ nonsplittable mono there exists g': $B \longrightarrow Y$ with $gu' = h'$.

THEOREM 3. *Assume Λ is an infinite artin algebra of finite representation type. Let*

$$0 \longrightarrow A \longrightarrow n_1B_1 \amalg n_2B_2 \amalg \cdots \amalg n_sB_s \longrightarrow A' \longrightarrow 0$$

be an almost split sequence in $\mathrm{mod}(\Lambda)$ *with B_i indecomposable, $B_i \not\cong B_j$ if $i \neq j$ and n_iB_i means the direct sum of n_i copies of B_i.*

(a) *$n_i \leqslant 3$ for every $i = 1, \ldots, s$.*

(b) *If for some i $n_i \geqslant 2$, then $n_j = 1$ if $j \neq i$.*

(c) *If Λ is a finite dimensional algebra over an algebraically closed field k, then $n_i = 1$ for any i.*

The idea of the proof is the following:

We can assume Λ indecomposable, denote by C the center of Λ. Then $K = C/\text{rad } C$ is a field. Now consider X and Y in mod(Λ). We define the set $I(X, Y) = \{\bar{f} \in \text{rad}(X, Y)/\text{rad}^2(X, Y) | f \text{ is an irreducible map}\}$. Now we put $K_X^* =$ units of End(X)/rad End(X), and the same for K_Y. In some cases $I(X, Y)$ is an affine K-variety and the K-algebraic group $K_X^* \times K_Y^*$ acts on $I(X, Y)$. Then using properties of irreducible maps [4] and the Gabriel-Tits argument [5] we get our theorem.

Following M. Auslander a skelletally small preadditive category $\mathcal{C}$ is said to be prevariety if: (a) Any object in $\mathcal{C}$ can be decomposed as finite sum of indecomposable objects in $\mathcal{C}$. (b) Any idempotent in $\mathcal{C}$ splits. (c) End(M) for any indecomposable object of $\mathcal{C}$ is a local ring. We recall that the Auslander graph $A(\mathcal{C})$ of $\mathcal{C}$ is defined as follows:

Choose a complete set of representatives M_i, $i \in I$, of all the isomorphism classes of indecomposable objects in $\mathcal{C}$. Then the vertices of $A(\mathcal{C})$ are the elements of I. We put an arrow from i to j iff there exists an irreducible map f: $M_i \longrightarrow M_j$.

Now we define the Auslander species of $\mathcal{C}$ by attaching to each point $i \in A(\mathcal{C})$ the division ring $K_i = \text{End}(M_i)/\text{rad End}(M_i)$, and to each arrow $i \longrightarrow j$ in $A(\mathcal{C})$ the $K_i - K_j$ bimodule $M_{ij} = \text{rad}(M_i, M_j)/\text{rad}^2(M_i, M_j)$.

We note that when Λ is an artin algebra and $\mathcal{C} = \text{pr}(\Lambda)$ then the Auslander species of $\mathcal{C}$ is just the Dlab-Ringel species of Λ (see [6]). As in [7] we can associate to the Auslander species of mod(Λ) a tensor category T_Λ.

We define $\text{rad}^i(X, Y)$ in similar way as $\text{rad}^2(X, Y)$ was defined. We put $\text{rad}^\infty(X, Y) = \bigcap_{i \geq 1} \text{rad}^i(X, Y)$. Here $\text{rad}^\infty(X, Y)$ is an ideal in mod(Λ). Then as in [7] we have the following:

PROPOSITION 2. *If Λ is either an hereditary artin algebra of finite representation type or a finite dimensional algebra over an algebraically closed field k, then there exists a full functor G: $T \longrightarrow$* mod(Λ)/rad$^\infty$ *such that for any indecomposable module M in* mod(Λ) *there exists M' in T_Λ with $G(M) \cong M'$.*

Observe that if Λ is of finite representation type than $\text{rad}^\infty(X, Y) = 0$ for any X and Y in mod(Λ).

Using properties of hereditary artin algebras proved in [2] we can describe Ker G in terms of almost split sequences.

REFERENCES

1. M. Auslander, *Representation theory of Artin algebras.* I, Comm. Algebra 1 (1974), 177–268.

2. M. Auslander and M. I. Platzeck, *Representation theory of hereditary Artin algebras,* Representation Theory of Algebras, (Proc. Conf. Temple University, Philadelphia, 1976, pp. 369–424). Lecture Notes in Pure and Applied Math., vol. 37, Dekker, New York, 1978.

3. M. Auslander and I. Reiten, *Representation theory of Artin Algebras.* III, *Almost split sequences,* Comm. Algebra 3 (1975), 239–294.

4. ———, *Representation theory of Artin algebras.* IV, *Invariants given by almost split sequences*, Comm. Algebra **5** (1977), 443–518.

5. I. N. Bernstein, I. M. Gelfand and V. A. Ponomariev, *Coxeter functors and Gabriel's theorem*, Uspeshi Mat. Nauk **28** (1973), 19–33.

6. V. Dlab and C. M. Ringel, *Indecomposable representation of graphs and algebras,* Mem. Amer. Math. Soc. **173** (1976).

7. ———, *On algebras of finite representation type*, J. Algebra **33** (1975), 306–394.

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