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COUNTEREXAMPLES TO GLOBAL TORELLI FOR CERTAIN SIMPLY CONNECTED SURFACES

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The question of whether or not one can distinguish birationally distinct surfaces by examining their period matrices is known as the Torelli problem. Let $\mathfrak{X} \xrightarrow{\pi} M$ be a proper holomorphic map such that each fibre is a nonsingular compact complex surface. As in [3], one can define the period mapping. Let U_0 be a sufficiently small neighborhood of $t_0 \in M$ and suppose that for all t', t'' in U_0, $t' \neq t''$, $S'_t = \pi^{-1}(t')$ is not birationally equivalent to $S_{t''}$. One aspect of the Torelli problem is: whether or not the period mapping, restricted to U_0, has a positive dimensional fibre at the point corresponding to the surface S_{t_0}.

In this short note we wish to announce the existence of simply connected surfaces for which the period mapping has positive dimensional fibres, therefore, for these surfaces the Torelli problem has a negative solution. However the surfaces we construct are rather special. In Theorem 1, the family of surfaces constructed has the numerical invariants $(c_1)^2 = 1, 2$, $p_g = 1$. The surfaces with $(c_1)^2 = 1$, $p_g = 1$ have also been studied by [1], [7], those with $(c_1)^2 = 2$, $p_g = 1$ by [2].

The other series of examples, given by Theorem 2, have the numerical invariants $(c_1)^2 = 0$, $p_g =$ any positive integer. They are simply connected elliptic surfaces, (see [4], [5], [6]) with one or two multiple fibres. Unlike the examples of Theorem one, these surfaces are generic points of their moduli spaces. In both cases, the canonical system exhibits unusual properties. It either consists of a single (unique) curve or has fixed components. If one looks at the moduli of the fixed components, as curves, then it seems likely the surfaces can be distinguished (locally). In general one must look at the mixed Hodge structure on the complement of either the unique canonical curve ($p_g = 1$) or the complement of the fixed components.

THEOREM 1. (a) *There exists a simply connected surface V_1, with*

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$(c_1)^2 = 1, p_g = 1$, *for which the period mapping has a positive dimensional fibre at the point corresponding to the surface* V_1.[1]

(b) *There exists a simply connected surface* V_2, *with* $(c_1)^2 = 2, p_g = 1$, *for which the period mapping has a positive dimensional fibre at the point corresponding to* V_2. *Furthermore, the surfaces constructed in this theorem all have a pencil of genus two curves.*

We will give a bare outline of the proof. The dimension of the period domain, for a regular surface with $p_g = 1$, is equal to the second Betti number minus 2. By Noether's formula, this is equal to 20 minus $(c_1)^2$. If this surface belongs to a family, each member having r independent algebraic cycles, then the image of this family in the period domain must have codimension at least r. In part (a), we construct a fifteen dimensional family of birationally distinct surfaces, each member of which has 5 independent algebraic cycles. The dimension of its image in the period domain, therefore, cannot be greater than $19 - 5 = 14$. To construct this family, let Σ be the rational ruled surface $\mathbf{P}_1 \times \mathbf{P}_1$. Let L_1 denote a generator of one ruling, L_2 the other. We construct a double covering of Σ ramified along a curve linearly equivalent to $6L_1 + 6L_2$, having as its only singularities three generic infinitely close triple points. Denote this surface by S, one calculates (see [9]) that $c_1^2(S) = 1, c_2(S) = 23$. By the proper choice of a branch curve, one can show that $\pi_1(S) = \{1\}$ and hence $p_g(S) = 1$. The dimension of the family of such surfaces can be shown to be equal to $48 - 27 - 6 = 15$. Each infinitely close triple point is at most 9 conditions and $\dim|6L_1 + 6L_2| = 48$. One can show that for generic S, all the surfaces in a sufficiently small neighborhood of the point corresponding to S are birationally distinct. The proof of part (b) is similar; see pp. 316–320 of [2] for their construction.

THEOREM 2. *Let S be a simply connected, minimal, elliptic surface with* $p_g(S) \neq 0$, *having one or at most two multiple fibres, then the period mapping has a positive dimensional fibre at the point corresponding to the surface S.* (*See* [4], [5], [6], [8].)

Again, we will give only a bare outline of the proof. It is known that: S is the logarithmic transform of B^η (see [4]); $S = S_{a,b,\eta} = L_a(m)L_b(n)(B^\eta)$, where, m and n are relatively prime positive integers with $m \neq 1$, B is an elliptic fibration with a section over the curve Δ, $a, b \in \Delta$, and $\eta \in H^1(\Delta, \Omega(B^\#))$. Let C_u^* denote the fibre of B^η over $u \in \Delta$, C_u denote the fibre of S over $u \in \Delta$. $S - C_a - C_b$ is isomorphic to $B^\eta - C_a^* - C_b^*$, and under this isomorphism one can regard a basis of holomorphic two forms on B^η as a basis for the homomorphic two forms on S. One shows: there exists a basis for $H_2(S, \mathbf{Z})$, $\delta_1, \ldots, \delta_k$, α, β, where β is algebraic, $\delta_i \cdot C_u = 0$, $\delta_i \cdot \alpha = 0 = \delta_i \cdot \beta$. Moreover, it can be

[1] It has been communicated to me that A. Todorov has also, recently, constructed a counterexample to global Torelli with these invariants.

shown that the only period that could change when one varies η and $a, b, \in \Delta$ is that due to the two cycle α. By an explicit integration, it can be further shown: that if a, b are moved little to a', b', then one can vary η so as to keep the periods, with respect to α unchanged. Moreover, the elliptic fibration can be recovered by examining the pluricanonical system. $S_{a',b',\eta'}$ is therefore not isomorphic to $S = S_{a,b,\eta}$, as can be seen by examining the location of the multiple fibres relative to the other singular fibres.

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