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A CHARACTERIZATION OF HEREDITARY RINGS OF FINITE REPRESENTATION TYPE

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In this note we announce a characterization of hereditary rings R of finite representation type in terms of a sequence of partial Coxeter functors associated to R.

The methods we use are theory of almost split sequences [1], partial Coxeter functors [2], [3], [5], representations of species [7], and some ideas of Ringel [8, §6].

We recall from [7] that a *species* $\mathsf{M} = (F_i, {}_iM_j)_{i,j \in I}$ is a finite set of division rings F_i and $F_i - F_j$ bimodules ${}_iM_j$. Throughout we will suppose M is a species such that ${}_iM_j \neq 0$ implies ${}_jM_i = 0$ for $i \neq j$. M is called *finite dimensional* if the dimensions

$$d_{ij} = \dim({}_iM_j)_{F_j}, \qquad d_{ji} = \dim_{F_i}({}_iM_j)$$

are finite for $i \neq j$. We denote by $\mathcal{R}(\mathsf{M})$ the category of all finite dimensional right representations of M (see [6], [8]). From M we derive an *oriented valued graph* $(\Gamma, \mathbf{d}, \Omega)$ (not necessarily symmetrizable) with valued edges

$$\underset{i}{\bullet} \xrightarrow{(d_{ij},\, d_{ji})} \underset{j}{\bullet}$$

precisely when ${}_iM_j \neq 0$. Given source (resp. sink) k in $(\Gamma, \mathbf{d}, \Omega)$ we define a species $\mathsf{M}^k = (F_i, {}_iN_j)_{i,j \in I}$ by taking

$${}_iN_j = \begin{cases} {}_kM_i^i = \operatorname{Hom}_{F_i}({}_kM_i, F_i) & \text{for } j = k \quad (\text{resp.} = {}_jM_k^j \text{ for } i = k) \\ {}_iM_j \quad \text{for} \quad j \neq k & (\text{resp. for } i \neq k). \end{cases}$$

If M is a finite dimensional species and k is a sink in $(\Gamma, \mathbf{d}, \Omega)$ we can define a pair of *Coxeter functors*

$$\mathcal{R}(\mathsf{M}) \underset{S_k^-}{\overset{S_k^+}{\rightleftarrows}} \mathcal{R}(\mathsf{M}^k)$$

having following properties:

(c_1^+) S_k^- is left adjoint to S_k^+.

(c_2^+) Suppose $\mathbf{X} = (X_i, {}_j\varphi_i)$ is indecomposable in $\mathcal{R}(\mathsf{M})$. Then

(i) $S_k^+\mathbf{X} = 0$ if and only if $\mathbf{X} \cong \mathbf{F}_k$, where $\mathbf{F}_k$ has F_k on kth coordinate and zeros otherwise,

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(ii) if $S_k^+\mathbf{X} = 0$ then there is a natural isomorphism $S_k^-S_k^+\mathbf{X} \simeq \mathbf{X}$ and $\mathbf{dim}\, S_k^+\mathbf{X} = s_k(\mathbf{dim}\,\mathbf{X})$ where $\mathbf{dim}\,\mathbf{X} = (\dim(X_i)_{F_i})_{i\in I}$ and $s_k\colon Z^\Gamma \longrightarrow Z^\Gamma$ is the corresponding reflection defined by using the matrix (d_{ij}).

(c_3^+) If $\mathbf{X}$ and $\mathbf{Y}$ are indecomposables in $\mathcal{R}(\mathcal{M})$ and $S_k^+\mathbf{X} \neq 0$, $S_k^+\mathbf{Y} \neq 0$ then there is a natural isomorphism $\mathrm{Hom}(\mathbf{X}, \mathbf{Y}) \cong \mathrm{Hom}(S_k^+\mathbf{X}, S_k^+\mathbf{Y})$.

(c_4^+) If $0 \longrightarrow \mathbf{X} \longrightarrow \mathbf{Y} \longrightarrow \mathbf{Z} \longrightarrow 0$ is an almost split sequence in $\mathcal{R}(\mathcal{M})$ with $\mathbf{X}$ noninjective and $S_k^+\mathbf{X} \neq 0$ then $0 \longrightarrow S_k^+\mathbf{X} \longrightarrow S_k^+\mathbf{Y} \longrightarrow S_k^+\mathbf{Z} \longrightarrow 0$ is almost split.

The properties (c_1^+)–(c_4^+) and analogous properties (c_1^-)–(c_4^-) for the case k is a source can be found in [2], [5].

Now let $\mathcal{M}$ be a finite dimensional species, $(\Gamma, \mathbf{d}, \Omega)$ its valued graph and $k_1, \ldots, k_n$ a fixed admissible orientation (see [5]). We define a sequence $\{k_j'\}_{j\in Z}$, where Z is the set of integral numbers. We put

$$k_j' = \begin{cases} k_{r+1} & \text{if } j \geqslant 0,\ j = tn + r,\ 0 \leqslant r < n, \\ k_{n-r+1} & \text{if } j < 0,\ -j = tn + r,\ 0 \leqslant r < n. \end{cases}$$

Moreover for $m \in Z$ we define inductively species $\mathcal{M}^{(m)}$ putting $\mathcal{M}^{(0)} = \mathcal{M}$ and

$$\mathcal{M}^{(m)} = \begin{cases} (\mathcal{M}^{(m-1)})^{k'_{m-1}} & \text{for } m \geqslant 1, \\ (\mathcal{M}^{(m+1)})^{k'_m} & \text{for } m \leqslant -1. \end{cases}$$

We say that $\mathcal{M}$ has *finite dimension property* if all species $\mathcal{M}^{(m)}$, $m \in Z$, are finite dimensional. In this case we denote by $(\Gamma, \mathbf{d}^m, \Omega^m)$ the valued graph of the species $\mathcal{M}^{(m)}$ and by $s_m\colon Z^\Gamma \longrightarrow Z^\Gamma$ the reflection $s_{k'_m}$ defined by using the matrix $\mathbf{d}^m = (d_{ij}^m)$. Following Ringel [8] we say that $\mathcal{M}$ has the *constant dimension property* if $d_{ij}^m = d_{ij}$ for all $i, j \in I$, $m \in Z$.

Now given a species $\mathcal{M}$ with the finite dimension property we have an infinite diagram

$$\begin{array}{ccccccccccccc}
\cdots \rightleftarrows & \mathcal{R}(\mathcal{M}^{(-2)}) & \underset{S_{-2}^-}{\overset{S_{-2}^+}{\rightleftarrows}} & \mathcal{R}(\mathcal{M}^{(-1)}) & \underset{S_{-1}^-}{\overset{S_{-1}^+}{\rightleftarrows}} & \mathcal{R}(\mathcal{M}) & \underset{S_1^-}{\overset{S_1^+}{\rightleftarrows}} & \mathcal{R}(\mathcal{M}^{(1)}) & \underset{S_2^-}{\overset{S_2^+}{\rightleftarrows}} & \mathcal{R}(\mathcal{M}^{(2)}) & \rightleftarrows \cdots \\
& \Big\downarrow \mathbf{dim} & & \Big\downarrow \mathbf{dim} & & \Big\downarrow \mathbf{dim} & & \Big\downarrow \mathbf{dim} & & \Big\downarrow \mathbf{dim} & \\
\cdots \rightleftarrows & Z^\Gamma & \underset{s_{-2}}{\overset{s_{-2}}{\rightleftarrows}} & Z^\Gamma & \underset{s_{-1}}{\overset{s_{-1}}{\rightleftarrows}} & Z^\Gamma & \underset{s_1}{\overset{s_1}{\rightleftarrows}} & Z^\Gamma & \underset{s_2}{\overset{s_2}{\rightleftarrows}} & Z^\Gamma & \rightleftarrows \cdots
\end{array}$$

where S_i^+ and S_i^- are appropriate partial Coxeter functors and s_i the corresponding reflections. The diagram will be called the *infinite Coxeter scheme* of $\mathcal{M}$ with respect to the admissible sequence $k_1, \ldots, k_n$. Now we are able to formulate the main result of the paper.

THEOREM. *Let $\mathcal{M}$ be a finite dimensional species and suppose that its valued graph $(\Gamma, \mathbf{d}, \Omega)$ is connected. The category $\mathcal{R}(\mathcal{M})$ is of finite representation type if and only if $\mathcal{M}$ has the finite dimension property and there exists a* *number* $m > 0$ *such that* $s_m \cdots s_1(e_i) \not> 0$ *for any source* i *of* Γ, *where* $e_i \in Z^\Gamma$ *is the vector having* 1 *on the i-th coordinate and zeros otherwise. Moreover, if m is minimal with the above property then*

(a) *The mapping* **dim**: $\mathfrak{R}(\mathsf{M}) \longrightarrow Z^\Gamma$ *is a one-one correspondence between isomorphism classes of indecomposable representations in* $\mathfrak{R}(\mathsf{M})$ *and vectors in* Z^Γ *of the form* $s_1 \cdots s_{t-1}(e_{k'_t})$ where $t < m$ and k'_t *is a sink in* $(\Gamma, \mathbf{d}^t, \Omega^t)$. *In other words any indecomposable representation* **X** *in* $\mathfrak{R}(\mathsf{M})$ *has the form* $\mathbf{X} \cong S_1^- \cdots S_{t-1}^- \mathbf{F}_{k'_t}$, *where* $t < m$ *and* $\mathbf{F}_{k'_t}$ *is a simple projective in* $\mathfrak{R}(\mathsf{M}^{(t-1)})$ *with* $F_{k'_t}$ *on the* k'_t*th coordinate and zeros otherwise.*

(b) *There are equivalences* $\mathfrak{R}(\mathsf{M}^{(i)}) \approx \mathfrak{R}(\mathsf{M}^{(m'q+i)})$ *for* $i = 0, 1, \ldots, m' - 1$ *and* $q \in Z$.

Observe that the theorem gives us a characterization of hereditary rings R of finite representation type having the property that R are generated over their centers by a set of cardinality $\aleph_\iota$, the first strongly inaccessible cardinal number. Indeed, if R is such a ring and M_R is the species of R then by [9, Corollary 4.6] there is an equivalence mod-$R \simeq \mathfrak{R}(\mathsf{M}_R)$.

Let us recall from [6] that if M is finite dimensional and the division rings F_i, $i \in I$, are finitely generated over their centers then M has the constant dimension property and therefore $\mathfrak{R}(\mathsf{M})$ is of finite representation type if and only if its valued graph is a Dynkin diagram.

REMARK. *There exists a hereditary ring R of infinite representation type such that its valued graph* $(\Gamma, \mathbf{d}, \Omega)$ *is the Dynkin diagram*

$$\cdot \xrightarrow{(1,2)} \cdot$$

(*apply* [2] *and* [4]).

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