

ON HURWITZ' "84($g - 1$) THEOREM" AND PSEUDOFREE ACTIONS

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1. Definitions. An *admissible* space is a finitistic space cf. [B, p. 133] with finitely generated integral homology. Let a group G act on a topological space X , and $\pi: X \rightarrow G \backslash X$ the corresponding projection onto the orbit space. Then $S = \bigcup_{g \in G - e} X^g$ is the *singular* set and $\pi(S)$ the *branch* set of the G -action. The G -action is said to be *free* resp. *semifree* resp. *pseudofree* if $S = \emptyset$ resp. $S = X^G$ resp. S is discrete. If X is admissible, G finite acting pseudofreely on X then S is finite. In this case for $P^* \in \pi(S)$ the order n_{P^*} of the isotropy subgroup of G at $P \in \pi^{-1}(P^*)$ is called the *branching index* of the action at P^* .

2. An extension of Hurwitz' theorem. Let $\text{Aut } X$ denote the full group of automorphisms of a Riemann surface X . If X is closed and has genus $g \geq 2$ Hurwitz cf. [H] proved that $|\text{Aut } X| \leq 84(g - 1)$ and that the action of $\text{Aut } X$ on the space of holomorphic differentials is faithful. Now every nonidentity holomorphic selfmap of X has only *isolated* fixed points and also $\text{Aut } X$ (since it leaves a Riemannian metric invariant) is a compact Lie group. This theorem then extends to arbitrary pseudofree actions as follows. Let $\mathbf{N} = \{1, 2, \dots\}$ and χ denote the Euler characteristic.

(2.1) THEOREM. *There exists $h: \mathbf{N} \rightarrow \mathbf{R}_{>0}$ with the following property. If X is admissible, $\chi(X) < 0$, $m(X) = \sum_{i \geq 0} \dim H_{2i}(X; \mathbf{Q})$ and G is a compact Lie group acting on X so that every finite subgroup acts pseudofreely. Then G is finite, $|G| \leq h(m(X))|\chi(X)|$ and the action of G on $H_*(X; \mathbf{Q})$ is faithful.*

EXAMPLES. One has $h(1) = 6$, $h(2) = 42$, $h(3) = 1806$, \dots . In Hurwitz' theorem one has $m(X) = 2$, $\chi(X) = 2 - 2g$ and so $h(2)|\chi(X)| = 84(g - 1)$. For all other closed or nonclosed surfaces with $\chi(X) < 0$ one has $m(X) = 1$ and the bound for G is $6|\chi(X)|$. In the case of a closed nonorientable surface U_h with $h \geq 3$ crosscaps this bound is $6(h - 2)$. As in the classical case cf. [M], [S] it may be shown that this bound is attained for infinitely many h 's. Concretely there exist pseudofree actions of the alternating groups A_4 , A_5 and the

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Klein's simple group of order 168 on U_4 , U_{12} and U_{30} resp.

3. The case $\chi(X) \geq 0$.

(3.1) THEOREM. *There exists a function $I: \mathbf{N} \cup \{0\} \rightarrow \mathbf{R}_{>0}$ with the following property. If X is admissible, $\chi(X) = 0$, $m(X) = \sum_{i \geq 0} \dim H_{2i}(X; \mathbb{Q})$ and G is a compact Lie group so that every finite subgroup of G acts pseudofreely, then the isotropy subgroups of G are finite with order bounded by $I(m(X) - 1)$ and act faithfully on $H_*(X; \mathbb{Q})$.*

In case X is admissible, $\chi(X) > 0$ and G a finite group acting pseudofreely on X we say that G is *exceptional* if (i) G acts trivially on $H_*(X; \mathbb{Q})$, (ii) G does not act semifreely and (iii) no partial sum of the reciprocals of the branching indices is an integer.

(3.2) THEOREM. *There exists $J: \mathbf{N} - \{1\} \rightarrow \mathbf{R}_{>0}$ with the following property. Let X be admissible and $\chi(X) > 0$. Then exceptional groups exist only if $\chi(X) \geq 2$ in which case their order is bounded by $J(\chi(X))$.*

4. Pseudofree actions on cohomology manifolds. We have classified finite groups admitting pseudofree actions on cohomology manifolds with some homology properties common with disks, spheres² and real and complex projective spaces. For the case of S^2 such result was proved by Smith cf. [Sm, pp. 407–409]. His proof unfortunately does not generalize to other cases. As a sample of these results I mention

(4.1) THEOREM. *Let X be a \mathbf{Z} -cohomology manifold of dimension $2d > 2$ with \mathbf{Z} -cohomology ring isomorphic to that of $\mathbf{P}_d(\mathbf{C})$ and G a finite group acting pseudofreely on X . Then (1) $d + 1 \neq \text{a prime} \Rightarrow G$ is cyclic, (2) $d + 1 = p$, a prime $\Rightarrow G$ is either (i) cyclic or (ii) $\mathbf{Z}_p \times \mathbf{Z}_p$ or (iii) $\mathbf{Z}_n \rtimes \mathbf{Z}_p$ where $p^2 \nmid n$ and \mathbf{Z}_p acts nontrivially on \mathbf{Z}_n .*

5. Remarks on proofs of (2.1), (3.1) and (3.2).

(5.1) THEOREM. *Let a finite group G act (not necessarily pseudofreely) on X and assume that for all subgroups H of G , X^H is admissible. Then*

- (i) $\sum_{g \in G} \chi(X^g) = |G| \chi(G \backslash X)$,
- (ii) $\chi(X) - \chi(S) = |G| \{ \chi(G \backslash X) - \chi(G \backslash S) \}$.

For $X =$ a finite complex these equations may be proved directly. In the general case we use [Bw], [Z] allowing an interpretation of $\chi(X^g)$ as the Lefschetz number $L(g)$. Roughly speaking these equations generalize the classical Riemann Hurwitz formula. The rest is an understanding and refinement of Hurwitz' original argument in [H].

²J. Shaneson informed me that he and S. Cappell have also classified groups acting pseudofreely on spheres in a sequel to [CS].

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