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ON SIMPLICITY OF CERTAIN INFINITE DIMENSIONAL LIE ALGEBRAS

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1. The main statements. Let $A = (a_{ij})$ be a complex $(n \times n)$-matrix. Denote by $\widetilde{\mathfrak{G}}(A)$ a complex Lie algebra with $3n$ generators $e_i, f_i, h_i, i \in I = \{1, \ldots, n\}$, and the following defining relations ($i, j \in I$):

$$[e_i, f_j] = \delta_{ij}h_i, \qquad [h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij}e_j, \qquad [h_i, f_j] = -a_{ij}f_j.$$

Set $\widetilde{C} = \{c_1h_1 + \cdots + c_nh_n | a_{1j}c_1 + \cdots + a_{nj}c_n = 0, j \in I\}$; clearly, $\widetilde{C}$ lies in the center of $\widetilde{\mathfrak{G}}(A)$. Set $\Gamma = \mathbf{Z}^n$, $\Gamma_+ = \{(k_1, \ldots, k_n) \in \Gamma | k_i \geqslant 0\} \backslash \{0\}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the standard basis of Γ. For $\eta = (k_1, \ldots, k_n)$ set $T_\eta = \Sigma_{i,j}\, a_{ij}k_ik_j - \Sigma_i\, a_{ii}k_i$.

THEOREM 1. *Provided that $a_{ij} = a_{ji}$, $i, j \in I$, and $T_\eta \neq 0$ for any $\eta \in \Gamma_+ \backslash \Pi$, the Lie algebra $\widetilde{\mathfrak{G}}(A)/\widetilde{C}$ is simple,*

COROLLARY 1. *Provided that A is a real symmetric matrix with positive entries, the Lie algebra $\widetilde{\mathfrak{G}}(A)/\widetilde{C}$ is simple.*

COROLLARY 2. *The Lie algebra K_2 with the generators e_1, e_2, f_1, f_2, h and the defining relations $[e_i, f_j] = \delta_{ij}h$, $[h, e_i] = e_i$, $[h, f_i] = -f_i$ is simple.*

PROOF.

$$K_2 = \widetilde{\mathfrak{G}}\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) / \widetilde{C}.$$

Corollary 2 has been conjectured in [1]. Further one can find a motivation for this problem. Setting $\deg e_i = -\deg f_i = \alpha_i$, $\deg h_i = 0$, $i \in I$, defines a Γ-gradation $\widetilde{\mathfrak{G}}(A) = \bigoplus_{\alpha \in \Gamma} \widetilde{\mathfrak{G}}_\alpha$. Let $\mathfrak{J}$ be the sum of all graded ideals in $\widetilde{\mathfrak{G}}(A)$ intersecting $\widetilde{\mathfrak{G}}_0$ trivially. We set $\mathfrak{G}(A) = \widetilde{\mathfrak{G}}(A)/\mathfrak{J}$; let $\mathfrak{G}(A) = \bigoplus_{\alpha \in \Gamma} \mathfrak{G}_\alpha$ be the induced gradation. Note that if D is a nondegenerate diagonal matrix, then $\mathfrak{G}(DA) \simeq \mathfrak{G}(A)$; the matrices A and DA are called *equivalent.* Let C be the image of $\widetilde{C}$ in $\mathfrak{G}(A)$; then C is the center of $\mathfrak{G}(A)$ [1]. The Lie algebra $\mathfrak{G}(A)/C$ has no graded ideals if and only if [2]

(m) for any $i, j \in I$ there exists $i_1, \ldots, i_r \in I$ such that $a_{ii_1}a_{i_1i_2} \cdots a_{i_rj} \neq 0$.

If A is the Cartan matrix of a simple finite dimensional Lie algebra $\mathfrak{G}$,

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then $\mathfrak{G} \simeq \mathfrak{G}(A)$. In general, the Lie algebras $\mathfrak{G}(A)$ are infinite dimensional. A number of applications of these algebras in various fields of mathematics have been found in the last decade. The Lie algebra K_2 plays the role of a "test" algebra in [1]. Due to the fact that K_2 is simple, we immediately obtain a stronger form of Theorem 1 from [1].

THEOREM 2. *Suppose that matrix A satisfies the condition* (m). *Then there are only the following three possibilities*:

(i) *A is equivalent to the Cartan matrix of a simple finite dimensional Lie algebra $\mathfrak{G}$ (and $\mathfrak{G}(A) \cong \mathfrak{G}$)*;

(ii) *A is equivalent to one of the matrices from Tables* 1–3 [1], *and the Gelfand-Kirillov dimension of $\mathfrak{G}(A)$ is* 1 (*the construction of $\mathfrak{G}(A)/C$ is given by Lemma* 22 [1]);

(iii) *$\mathfrak{G}(A)$ contains a free subalgebra of rank* 2 *and the Lie algebra $\mathfrak{G}(A)/C$ is simple.*

Suppose that the matrix A is symmetric. Then there exists an invariant symmetric bilinear form (,) on $\widetilde{\mathfrak{G}}(A)$ which is uniquely defined by the properties (a) $(h_i, h_j) = a_{ij}$ and $(e_i, f_j) = \delta_{ij}$, $i, j \in I$, (b) $(\widetilde{\mathfrak{G}}_\alpha, \widetilde{\mathfrak{G}}_\beta) = 0$ for $\alpha \neq -\beta$, (c) $\operatorname{Ker}(,) = \mathfrak{J} + \widetilde{C}$ [1]. Let σ be an involutive antiautomorphism of $\widetilde{\mathfrak{G}}(A)$ defined by $\sigma(e_i) = f_i$, $\sigma(f_i) = e_i$, $\sigma(h_i) = h_i$. On each $\widetilde{\mathfrak{G}}_\alpha$, $\alpha \in \Gamma_+$, we introduce a bilinear form by $B_\alpha(x, y) = (x, \sigma(y))$, $x, y \in \widetilde{\mathfrak{G}}_\alpha$. Since $\bigoplus_{\alpha\in\Gamma_+} \mathfrak{G}_\alpha$ is freely generated by $e_1, \ldots, e_n$, we can fix a basis in each $\widetilde{\mathfrak{G}}_\alpha$ which does not depend on A. Let $\varphi_\alpha = \varphi_\alpha(A)$ be the determinant of the matrix of B_α in this basis. This is a function on the space of symmetric $(n \times n)$-matrices. It follows from Theorem 1 that provided that $T_\eta \neq 0$ for any $\eta \in \Gamma_+ \backslash \Pi$, the Lie algebra $\widetilde{\mathfrak{G}}(A)/\widetilde{C}$ is simple. Hence, φ_α is different from 0 outside the hyperplanes $T_\eta = 0$, $\eta \in \Gamma_+ \backslash \Pi$, and we obtain

THEOREM 3. *Up to a nonzero constant factor* (*depending on the basis*) *one has*:

$$\varphi_\alpha(A) = \prod_{\eta\in\Gamma_+\backslash\Pi} T_\eta^{c_{\eta,\alpha}},$$

where $c_{\eta,\alpha}$ are nonnegative integers.

REMARK. An interesting open question is to compute the exponents $c_{\eta,\alpha}$. It follows from the proof of Theorem 1 that $c_{\eta,\alpha} = 0$ if $\alpha = k\alpha_i$ or $\alpha - \eta \notin \Gamma_+ \cup \{0\}$. It is also clear that $\deg \varphi_\alpha = (\text{height } \alpha - 1) \dim \widetilde{\mathfrak{G}}_\alpha$.

2. **Proof of Theorem 1.** Set $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Gamma_+} \mathfrak{G}_{\pm\alpha}$ and $\mathfrak{H} = \mathfrak{G}_0$; then $\mathfrak{G}(A) = \mathfrak{n}_- \otimes \mathfrak{H} \oplus \mathfrak{n}_+$. Since $\mathfrak{G}(A)/C$ is simple ([1, Lemma 6]) the theorem will follow from the fact that $\mathfrak{n}_-$ is a free Lie algebra with free generators $f_1, \ldots, f_n$. To prove this, we employ the *highest weight representations* $M(\lambda)$, $\lambda \in \mathfrak{H}^*$, of

$\mathfrak{G}(A)$ [3]. We recall that $M(\lambda) = U(\mathfrak{G}(A)) \otimes_{U(\mathfrak{H} \oplus \mathfrak{n}_+)} \mathbf{C}_\lambda$, where $\mathbf{C}_\lambda$ is a 1-dimensional representation defined by $\mathfrak{n}_+(1) = 0$, $h(1) = \lambda(h)$, $h \in \mathfrak{H}$. The gradation of $\mathfrak{G}(A)$ induces a gradation: $M(\lambda) = \bigoplus_{\eta \in \Gamma_+ \cup \{0\}} M(\lambda)_{-\eta}$. We set $\operatorname{ch} M(\lambda) = e^\lambda \Sigma_\eta (\dim M(\lambda)_{-\eta}) e^{-\eta}$. Clearly one has

$$\operatorname{ch} M(\lambda) = e^\lambda \prod_{\alpha \in \Gamma_+} (1 - e^{-\alpha})^{-\dim \mathfrak{G}_{-\alpha}}.$$

From now on we will assume that A is symmetric. We recall the definition of the *Casimir operator* Ω on the space $M(\lambda)$ (in a slightly modified form, cf. [3]). The form (,) on $\widetilde{\mathfrak{G}}(A)$ induces a bilinear form on $\mathfrak{G}(A)$ which we also denote by (,). Note that (,) is nondegenerate on $\mathfrak{G}_\alpha \oplus \mathfrak{G}_{-\alpha}$, $\alpha \in \Gamma_+$. We define a bilinear form on Γ by setting $(\alpha_i, \alpha_j) = a_{ij}$; we set $h_\eta = \Sigma\, k_i h_i$ for $\eta = \Sigma\, k_i \alpha_i$. We choose in each $\mathfrak{G}_\alpha$, $\alpha \in \Gamma_+$, a basis $e_\alpha^{(i)}$, $i = 1, \ldots, \dim \mathfrak{G}_\alpha$, and in $\mathfrak{G}_{-\alpha}$ a dual basis $e_{-\alpha}^{(i)}$. We define $\rho \in \mathfrak{H}^*$ by $\rho(h_i) = 1/2a_{ii}$, $i \in I$. Finally, we define Ω as follows:

$$\Omega(v) = ((\eta, \eta) - 2(\lambda + \rho)(h_\eta))v + 2 \sum_{\alpha \in \Gamma_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}(v), \; v \in M(\lambda)_{-\eta}.$$

A direct verification (cf. [4, Proposition 2.7]) shows that $\Omega = 0$. This and the fact that $M(\lambda)$ is irreducible if any vector killed by all $\mathfrak{G}_\alpha$, $\alpha \in \Gamma_+$, lies in $M(\lambda)_0$, gives the following lemma (see [5] for a more precise statement).

LEMMA 1. *If* $(\eta, \eta) - 2(\lambda + \rho)(h_\eta) \neq 0$ *for any* $\eta \in \Gamma_+$, *then the* $\mathfrak{G}(A)$-*module* $M(\lambda)$ *is irreducible.*

Now we are able to complete the proof of Theorem 1. Consider the $\mathfrak{G}(A)$-module $M = M(0)$. The module M contains submodules $L_i = U(\mathfrak{G}(A))(M(0)_{-\alpha_i})$; set $L = \Sigma_i L_i$. Clearly, $\dim M/L = 1$ and the $\mathfrak{G}(A)$-module L_i is isomorphic to $M(-\alpha_i)$. Moreover, since $(\eta, \eta) - 2(\rho - \alpha_i)(h_\eta) = T_{\eta + \alpha_i}$, by Lemma 1, $M(-\alpha_i)$ is irreducible and therefore L is a direct sum of L_i's. Hence, we have $\operatorname{ch} M/L = 1 = \operatorname{ch} M(0) - \Sigma_i \operatorname{ch} M(-\alpha_i)$. This gives the following formula:

$$\prod_{\alpha \in \Gamma_+} (1 - e^{-\alpha})^{\dim \mathfrak{G}_{-\alpha}} = 1 - \sum_{i=1}^{n} e^{-\alpha_i}. \tag{1}$$

But (1) is equivalent to the fact that $\mathfrak{n}_-$ is freely generated by f_i, $i \in I$ (indeed, the inverse of the left-hand side of (1) is the generating function of $U(\mathfrak{n}_-)$; but $\mathfrak{n}_-$ is free $\longleftrightarrow$ $U(\mathfrak{n}_-)$ is free [6] $\longleftrightarrow$ the generating function of $U(\mathfrak{n}_-)$ is the inverse of the right-hand side of (1)).

PROOF OF THEOREM 2. It follows from §II 6 of [1] (see also [2, Lemma 3.11]) that each time when A is not one of the matrices of (i) or (ii), the Lie algebra $\mathfrak{G}(A)$ contains K_2 and therefore (by Theorem 1) contains a free subalgebra of rank 2.

REMARK. The problem about the defining relations for arbitrary $\mathfrak{G}(A)$ is still open; the first unclear case is $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ (see the conjecture in [1, §II 7]). I think that this problem can be solved by a detailed study of the functions φ_α.

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