

THE CELL-LIKE MAPPING PROBLEM

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A major unresolved issue in topology is whether or not there is a dimension raising cell-like mapping. A *cell-like map* $f: X \rightarrow Y$ is a proper mapping between metrizable spaces such that $f^{-1}(y)$ has the shape of a point for each $y \in Y$ (i.e., every map of $f^{-1}(y)$ to a polyhedron is null-homotopic or, equivalently for finite dimensional X , $f^{-1}(y)$ admits a cellular embedding in a Euclidean space).

During the period of his pioneering work on cellular decompositions of 3-space, Bing asked if the quotient space of such a decomposition is an ANR [Bi]. In the mid-sixties, Armentrout asked if the quotient space of a cellular decomposition of E^3 is finite dimensional [Ar]. It was shown in [Koz-1] that these two questions are equivalent and both are settled affirmatively by the following theorem.

THEOREM. *If $f: X \rightarrow Y$ is a cell-like mapping defined on a subset of a 3-manifold, then $\dim Y \leq 3$.*

The theorem is best understood by considering the case with X a compact subset of E^3 . The Sphere Theorem easily implies that (1) a cell-like subset of E^3 has arbitrarily small aspherical open neighborhoods and (2) each component of the intersection of two aspherical open subsets of E^3 is aspherical. In [Koz-2], the cell-like mapping problem is reduced to an extension problem for maps into polyhedra; this extension problem is solvable for maps into aspherical polyhedra. These facts are combined to prove the theorem by producing an ϵ -map of Y into E^3 for each $\epsilon > 0$.

It was known previously that cell-like maps on 1-dimensional spaces do not raise dimension. This is proved by using the Vietoris Mapping Theorem to conclude that the image has (integral) cohomological dimension at most 1 and then appealing to the fact that covering dimension agrees with cohomological dimension in this case (both are "classified" by extensions of maps into S^1). The precise connection between the cell-like mapping problem and the classical question of whether or not covering dimension and cohomological dimension agree for compacta was recently established by R. D. Edwards [Ed-1]; he announced that a compactum with cohomological dimension $\leq n$ is the cell-like image of a compactum with covering dimension $\leq n$.

It seems appropriate to sketch two reductions of the cell-like mapping problem for such maps on ANR's and manifolds. First, the result of Bothe [Bo]

Received by the editors June 1, 1979.

AMS (MOS) subject classifications (1970). Primary 54F45, 57A10; Secondary 54C55, 54A35.

Key words and phrases. Cell-like, aspherical, 3-manifolds.

¹Research supported in part by an NSF grant.

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0002-9904/80/0000-0106/\$01.50

that each n -dimensional compactum imbeds in a $(n + 1)$ -dimensional ANR and the result of Sieklucki [Si] that a collection of pairwise disjoint n -dimensional closed subsets of an n -dimensional ANR is countable can be used to show that the cell-like mapping problem for n -dimensional compacta is equivalent to the cell-like mapping problem for $(n + 1)$ -dimensional ANR's. Second, using techniques from [Ed-2], R. Daverman has shown that, for $n \geq 4$, if there is a cell-like dimension raising mapping on an n -manifold, then there is a cell-like dimension raising mapping on an $(n - 2)$ -dimensional compactum. Starting with a cell-like mapping $f: M^n \rightarrow Y$ ($n \geq 4$), Daverman constructs a cell-like mapping $\bar{f}: M^n \rightarrow Y$ with \bar{f} one-to-one on an "infinite 1-skeleton" of M^n ; if $\dim Y = \infty$, then there is a compact subset $Y' \subseteq Y$ with $\dim Y' = \infty$ and with $\dim f^{-1}(Y') \leq n - 2$.

The following illustrate that even with substantial additional hypotheses there are currently few results. In [Dy] Dyer showed that if $f: X \rightarrow Y$ is an open mapping with $\dim X = n$ and with each $f^{-1}(y)$ a point or an arc, then $\dim Y = n$ or $n - 1$ provided Y is assumed to be finite dimensional. In [Bor] Borsuk pointed out that a consequence of Smale's Vietoris mapping theorem for homotopy [Sm] is that if a space Y is the image of ANR X under a proper map $f: X \rightarrow Y$ with each $f^{-1}(y)$ an ANR, then Y is an ANR provided Y is assumed to be finite dimensional. Both Dyer and Borsuk asked whether or not the assumption of finite dimensionality is necessary; these questions remain unsettled.

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