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ON THE UNION OF SETS OF SYNTHESIS AND DITKIN'S CONDITION IN REGULAR BANACH ALGEBRAS

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The questions have been raised, whether for every commutative semisimple regular Banach algebra with unit, the union of two sets of synthesis is again a set of synthesis, whether every set of synthesis is a Ditkin set, and whether a singleton which is of synthesis is a Ditkin set (cf. [4, pp. 31, 34, 41] and [2, p. 225]). The purpose of this note is to show that for the Banach algebra obtained by adjoining a unit to the Banach algebra introduced by Mirkil in [3], the answer to all three questions is in the negative. In fact Mirkil's proof of his main result [3, §5] contains implicitly a negative answer to the third (and therefore also to the second) question. Mirkils algebra **M** is defined as follows: Let **T** denote the circle group identified with the interval $[-\pi, \pi)$, the group operation being addition modulo 2π. **M** is the convolution algebra of all functions in $L^2(\mathbf{T})$ which are continuous on the interval $[-\pi/2, \pi/2]$ with norm defined by

$$\|f\| = \|f\|_2 + \sup\{|f(t)|: \ |t| \leqslant \pi/2\}.$$

(We recall that the convolution of two functions, f, g in $L^2(\mathbf{T})$ is defined by

$$f * g(t) = \frac{1}{2\pi}\int_{\mathbf{T}} f(t-x)g(x)\,dx, \quad t \in T.)$$

As shown in [3], **M** is a Banach algebra in which the trigonometric polynomials are dense, whose regular maximal ideal space can be identified with the set of integers **Z**, and its Gelfand representation is given by the Fourier transform $f \longrightarrow \hat{f}$ where $\hat{f}(n) = (1/2\pi)\int_{\mathbf{T}} f(x)e^{-inx}dx$, $n \in \mathbf{Z}$.

We shall denote by $\mathbf{M}_1$ the Banach algebra obtained by the standard adjunction of unit to **M** ($\mathbf{M}_1$ can also be regarded as the convolution algebra $M \oplus \mathbf{D}$ where **D** is the one dimensional vector space spanned by the unit point measure concentrated at τ). The maximal ideal space of $\mathbf{M}_1$ can be identified with $\mathbf{Z} \cup \{\infty\}$, the one point compactification of **Z**.

We refer to [2, p. 230] for the definitions of sets of synthesis for regular Banach algebras (see also [4, p. 28] where these sets are called Wiener sets). We shall give here the definition directly for the algebras **M** and $\mathbf{M}_1$.

Let $\mathbf{M}^*$ denote the dual space of **M**. For every ν in M^* we set $\hat{\nu}(n) =$

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$\overline{\langle e^{inx}, \nu \rangle}$, $n \in \mathbf{Z}$, and define the spectrum of ν to be the set

$$\sigma(\nu) = \{n \in \mathbf{Z}: \hat{\nu}(n) \neq 0\}.$$

Every function g in $L^2(\mathbf{T})$ defines an element of $\mathbf{M}^*$ by the mapping

$$f \longrightarrow \langle f, g \rangle = \frac{1}{2\pi} \int_{\mathbf{T}} f(x)\overline{g(x)}\, dx, \qquad f \in \mathbf{M}.$$

We shall denote this element also by g.

Definition 1. A set $E \subset \mathbf{Z}$ is called a set of synthesis for $\mathbf{M}$ if every element in $\mathbf{M}^*$ with spectrum contained in E, is in the w^* closure of the vector space spanned in $\mathbf{M}^*$ by the set $\{e^{inx}\colon\ n \in E\}$. By the Hahn-Banach theorem, this is equivalent to the requirement that $\langle f, \nu \rangle = 0$ for every pair $\nu \in \mathbf{M}^*$ and $f \in \mathbf{M}$ such $\sigma(\nu) \subset E$ and $\hat{f} = 0$ on E.

Definition 2. A closed set $K \subset \mathbf{Z} \cup \{\infty\}$ is called a set of synthesis for $\mathbf{M}_1$ if the set $K \cap \mathbf{Z}$ is of synthesis for $\mathbf{M}$.

Since the trigonometric polynomials are dense in $\mathbf{M}$, the empty set is of synthesis for $\mathbf{M}$, and therefore by Definition 2, the singleton $\{\infty\}$ is of synthesis for $\mathbf{M}_1$. (It is easy to see that the usual definition of sets of synthesis for regular Banach algebras given in [2, p. 230] coincides, for the algebra $\mathbf{M}_1$, with Definition 2.)

It is clear that the algebra $\mathbf{M}$ is semisimple and regular and therefore the same is true for the algebra $\mathbf{M}_1$.

We refer to [4, p. 30] for the definition of Ditkin sets (called there Wiener-Ditkin sets) for regular Banach algebras. We only note here that the condition for $\{\infty\}$ to be a Ditkin set for $\mathbf{M}_1$, is equivalent to the requirement that for every function f in $\mathbf{M}$ there exists a sequence of trigonometric polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n * f \longrightarrow f$ in the norm of $\mathbf{M}$.

In what follows, we denote for every pair of integers r and s by $r\mathbf{Z} + s$ the set of all integers of the form $rn + s$, $n \in \mathbf{Z}$. The negative answers to the questions mentioned, is contained in the following.

Theorem. (a) *The sets* $K_1 = 4\mathbf{Z} \cup \{\infty\}$ *and* $K_2 = 4\mathbf{Z} + 2 \cup \{\infty\}$ *are of synthesis for* $\mathbf{M}_1$ *but the set* $K_1 \cup K_2$ *is not of synthesis for* $\mathbf{M}_1$.

(b) *The singleton* $\{\infty\}$ *is not a Ditkin set for* $\mathbf{M}_1$.

Proof of (a). We have to show that the sets $E_1 = 4\mathbf{Z}$ and $E_2 = 4\mathbf{Z} + 2$ are of synthesis for $\mathbf{M}$ but the set $2\mathbf{Z} = E_1 \cup E_2$ is not of synthesis for $\mathbf{M}$. For this we need first to identify the dual space $\mathbf{M}^*$. Let $C[-\pi/2, \pi/2]$ denote as usual the Banach space of complex continuous functions on $[-\pi, \pi/2]$ with the sup norm, and consider the Banach space $\mathbf{B} = L^2(\mathbf{T}) \times C[-\pi/2, \pi/2]$ with norm $\|(f, h)\| = \|f\|_2 + \|h\|_{C[-\pi/2,\pi/2]}$. Noticing that the mapping

$$f \longrightarrow (f, f_{|[-\pi/2,\pi/2]}), \qquad f \in \mathbf{M}$$

(where $f_{|[-\pi/2,\pi/2]}$ denotes the restriction of f to $[-\pi/2, \pi/2]$) is an isometric isomorphism of **M** onto a closed subspace of **B**, using the Hahn Banach theorem, the fact that $\mathbf{B}^* = (L^2(\mathbf{T}))^* \times C^*[-\pi/2, \pi/2]$, and the known representations of the dual spaces of $L^2(\mathbf{T})$ and $C[-\pi/2, \pi/2]$, we see that every element ν in $\mathbf{M}^*$ can be represented by a Borel measure on **T** (which we also denote by ν) which admits a decomposition of the form $\nu = gdx + \mu$ where $g \in L^2(\mathbf{T})$ and μ is a Borel measure supported on $[-\pi/2, \pi/2]$, and the action of ν on **M** is given by

$$\langle f, \nu\rangle = \frac{1}{2\pi} \int_{\mathbf{T}} f(x)\overline{g(x)}\, dx + \int_{\mathbf{T}} f(x)\, d\bar{\mu}(x), \qquad f \in \mathbf{M}.$$

Clearly $\|\nu\|_{\mathbf{M}^*} \leqslant \max\{\|g\|_2, |\mu|(\mathbf{T})\}$ where $|\mu|(\mathbf{T})$ is the total variation of μ. Suppose now that $\nu = gdx + \mu \in \mathbf{M}^*$ with g and μ as described above, and assume that $\sigma(\nu) \subset E_1 = 4\mathbf{Z}$. Then $\hat{\nu}(n) = \int_{\mathbf{T}} e^{-int} d\nu(t) = 0$ for $n \notin 4\mathbf{Z}$ and therefore for every trigonometric polynomial p

$$\int_{\mathbf{T}} p(t + \tfrac{\pi}{2})\, d\nu(t) = \int_{\mathbf{T}} p(t)\, d\nu(t),$$

and since a continuous function on **T** is the uniform limit of trigonometric polynomials, the equality remains true if p is replaced by any such function. This shows that ν is a measure of period $\pi/2$, that is, $\nu(S + \pi/2) = \nu(S)$ for every Borel set $S \subset \mathbf{T}$. On the other hand, since μ is supported on $[-\pi/2, \pi/2]$, $\nu(S) = \int_S \overline{g(x)}\, dx$ for every Borel set $S \subset [0, \pi/2)$. Combining these facts we see that ν is in $L^2(\mathbf{T})$, and therefore setting $S_N(\nu) = \sum_{n=-N}^{N} \hat{\nu}(n) e^{inx}$, $N = 0, 1, \ldots,$ we deduce that

$$\|\nu - S_N(\nu)\|_{\mathbf{M}^*} \leqslant \|\nu - S_N(\nu)\|_2 \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

This shows that every element in $\mathbf{M}^*$ with spectrum contained in E_1, is even in the norm closure in $\mathbf{M}^*$, of the vector space spanned in $\mathbf{M}^*$ by the set $\{e^{inx}: n \in E_1\}$, and therefore E_1 is of synthesis for **M**. To show that $E_2 = 4\mathbf{Z} + 2$ is also a set of synthesis for **M**, consider a measure ρ in $\mathbf{M}^*$ with spectrum in E_2; then the spectrum of the measure $e^{2ix}\rho$ is in E_1, hence by the previous part of the proof this measure is in $L^2(\mathbf{T})$, and therefore ρ is also in $L^2(\mathbf{T})$. This implies as before, that E_2 is of synthesis for **M** (even in the norm topology of $\mathbf{M}^*$). The remaining assertions of the theorem follow from the main result in [3]. For the sake of completeness we include here a different proof which is more in line with the approach of this paper. To show that the set $E = E_1 \cup E_2$ is not of synthesis for **M**, consider the element of $\mathbf{M}^*$ defined by the measure $\mu = \delta_{\pi/2} + \delta_{-\pi/2}$ (for every $\tau \in \mathbf{T}$, δ_τ denotes the unit point measure concentrated at τ) and the function f in **M** defined by: $f(x) = 1$ for $|x| \leqslant \pi/2$ and $f(x) = -1$ for $\pi/2 < |x| \leqslant \pi$. Then $\hat{\mu}(n) = 2\cos(n\pi/2)$, $\forall\, n \in \mathbf{Z}$, $\hat{f}(n) = 2\sin(n\pi/2)/n\pi$, for $n \in \mathbf{Z}\setminus\{0\}$, and $\hat{f}(0) = 0$. Thus $\sigma(\mu) \subset E$ and $\hat{f} = 0$ on E, but $\langle f, \mu\rangle = 2$, and therefore E is not a set of synthesis for M.

PROOF OF (b). To show that $\{\infty\}$ is not a Ditkin set for $\mathbf{M}_1$, consider again the measure μ and the function f defined above. Noticing that $\langle q, \mu\rangle = 0$ for every trigonometric polynomial q such that $\hat{q} = 0$ on $2\mathbf{Z}$, we obtain for every trigonometric polynomial p (by using the identity $p * f = \Sigma_{n=-N}^{N}\hat{p}(n)\hat{f}(n)e^{inx}$, where N is the degree of p) that

$$\|f - p * f\|_{\mathbf{M}} \geqslant \|\mu\|_{\mathbf{M}^*}^{-1}\langle f - p * f, \mu\rangle = 1$$

and consequently $\{\infty\}$ is not a Ditkin set for $\mathbf{M}_1$.

REMARK. The proof of the theorem shows that the sets K_1 and K_2 are of synthesis for $\mathbf{M}_1$ even in the norm topology of $\mathbf{M}_1^*$ but $K_1 \cup K_2$ is not of synthesis for $\mathbf{M}_1$ even in the w^* topology of M_1.

The answers to the first and second questions mentioned at the beginning are not known for group algebras of locally compact noncompact abelian groups; in particular they are not known for the group algebra of $\mathbf{Z}$. A discussion of these problems and partial results can be found in [1, Chapter 1 and 2], [4, Chapters 2 and 6] and [5, Chapter 7].

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