ON THE UNION OF SETS OF SYNTHESIS
AND DITKIN'S CONDITION IN REGULAR BANACH ALGEBRAS

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The questions have been raised, whether for every commutative semisimple regular Banach algebra with unit, the union of two sets of synthesis is again a set of synthesis, whether every set of synthesis is a Ditkin set, and whether a singleton which is of synthesis is a Ditkin set (cf. [4, pp. 31, 34, 41] and [2, p. 225]). The purpose of this note is to show that for the Banach algebra obtained by adjoining a unit to the Banach algebra introduced by Mirkil in [3], the answer to all three questions is in the negative. In fact Mirkil’s proof of his main result [3, §5] contains implicitly a negative answer to the third (and therefore also to the second) question. Mirkil's algebra $M$ is defined as follows: Let $T$ denote the circle group identified with the interval $[-\pi, \pi)$, the group operation being addition modulo $2\pi$. $M$ is the convolution algebra of all functions in $L^2(T)$ which are continuous on the interval $[-\pi/2, \pi/2]$ with norm defined by

$$\|f\| = \|f\|_2 + \sup \{|f(t)|: |t| \leq \pi/2\}.$$ 

(We recall that the convolution of two functions, $f, g$ in $L^2(T)$ is defined by

$$f \ast g(t) = \frac{1}{2\pi} \int_T f(t-x)g(x) \, dx, \ t \in T.$$ )

As shown in [3], $M$ is a Banach algebra in which the trigonometric polynomials are dense, whose regular maximal ideal space can be identified with the set of integers $\mathbb{Z}$, and its Gelfand representation is given by the Fourier transform $f \mapsto \hat{f}$ where $\hat{f}(n) = (1/2\pi) \int_T f(x)e^{-inx} \, dx, n \in \mathbb{Z}$.

We shall denote by $M_1$ the Banach algebra obtained by the standard adjunction of unit to $M$ ($M_1$ can also be regarded as the convolution algebra $M \oplus D$ where $D$ is the one dimensional vector space spanned by the unit point measure concentrated at $\tau$). The maximal ideal space of $M_1$ can be identified with $\mathbb{Z} \cup \{\infty\}$, the one point compactification of $\mathbb{Z}$.

We refer to [2, p. 230] for the definitions of sets of synthesis for regular Banach algebras (see also [4, p. 28] where these sets are called Wiener sets). We shall give here the definition directly for the algebras $M$ and $M_1$.

Let $M^\ast$ denote the dual space of $M$. For every $\nu$ in $M^\ast$ we set $\hat{\nu}(n) =$
\( \langle e^{inx}, \nu \rangle, n \in \mathbb{Z} \), and define the spectrum of \( \nu \) to be the set 
\[
\sigma(\nu) = \{ n \in \mathbb{Z}; \hat{\nu}(n) \neq 0 \}.
\]
Every function \( g \) in \( L^2(T) \) defines an element of \( M^* \) by the mapping 
\[
f \mapsto (f, g) = \frac{1}{2\pi} \int_T f(x)g(x) \, dx, \quad f \in M.
\]
We shall denote this element also by \( g \).

**Definition 1.** A set \( E \subseteq \mathbb{Z} \) is called a set of synthesis for \( M \) if every element in \( M^* \) with spectrum contained in \( E \), is in the \( w^* \) closure of the vector space spanned in \( M^* \) by the set \( \{ e^{inx}; n \in E \} \). By the Hahn-Banach theorem, this is equivalent to the requirement that \( \langle f, \nu \rangle = 0 \) for every pair \( \nu \in M^* \) and \( f \in M \) such \( \sigma(\nu) \subseteq E \) and \( \hat{f} = 0 \) on \( E \).

**Definition 2.** A closed set \( K \subseteq \mathbb{Z} \cup \{ \infty \} \) is called a set of synthesis for \( M_1 \) if the set \( K \cap \mathbb{Z} \) is of synthesis for \( M \).

Since the trigonometric polynomials are dense in \( M \), the empty set is of synthesis for \( M \), and therefore by Definition 2, the singleton \( \{ \infty \} \) is of synthesis for \( M_1 \). (It is easy to see that the usual definition of sets of synthesis for regular Banach algebras given in [2, p. 230] coincides, for the algebra \( M_1 \), with Definition 2.)

It is clear that the algebra \( M \) is semisimple and regular and therefore the same is true for the algebra \( M_1 \).

We refer to [4, p. 30] for the definition of Ditkin sets (called there Wiener-Ditkin sets) for regular Banach algebras. We only note here that the condition for \( \{ \infty \} \) to be a Ditkin set for \( M_1 \), is equivalent to the requirement that for every function \( f \) in \( M \) there exists a sequence of trigonometric polynomials \( (p_n)_{n=1}^\infty \) such that \( p_n * f \longrightarrow f \) in the norm of \( M \).

In what follows, we denote for every pair of integers \( r \) and \( s \) by \( r\mathbb{Z} + s \) the set of all integers of the form \( rn + s, n \in \mathbb{Z} \). The negative answers to the questions mentioned, is contained in the following.

**Theorem.** (a) The sets \( K_1 = 4\mathbb{Z} \cup \{ \infty \} \) and \( K_2 = 4\mathbb{Z} + 2 \cup \{ \infty \} \) are of synthesis for \( M_1 \) but the set \( K_1 \cup K_2 \) is not of synthesis for \( M_1 \).

(b) The singleton \( \{ \infty \} \) is not a Ditkin set for \( M_1 \).

**Proof of (a).** We have to show that the sets \( E_1 = 4\mathbb{Z} \) and \( E_2 = 4\mathbb{Z} + 2 \) are of synthesis for \( M \) but the set \( 2\mathbb{Z} = E_1 \cup E_2 \) is not of synthesis for \( M \). For this we need first to identify the dual space \( M^* \). Let \( C[-\pi/2, \pi/2] \) denote as usual the Banach space of complex continuous functions on \( [-\pi, \pi] \) with the sup norm, and consider the Banach space \( B = L^2(T) \times C[-\pi/2, \pi/2] \) with norm \( \| (f, h) \| = \| f \|_2 + \| h \|_{C[-\pi/2,\pi/2]} \). Noticing that the mapping 
\[
f \mapsto (f, f|_{[-\pi/2,\pi/2]}), \quad f \in M
\]
where \( f_{[-\pi/2, \pi/2]} \) denotes the restriction of \( f \) to \([-\pi/2, \pi/2]\) is an isometric isomorphism of \( M \) onto a closed subspace of \( B \), using the Hahn Banach theorem, the fact that \( B^* = (L^2(T))^* \times C^*[-\pi/2, \pi/2] \), and the known representations of the dual spaces of \( L^2(T) \) and \( C[-\pi/2, \pi/2] \), we see that every element \( \nu \) in \( M^* \) can be represented by a Borel measure on \( T \) (which we also denote by \( \nu \)) which admits a decomposition of the form \( \nu = gdx + \mu \) where \( g \in L^2(T) \) and \( \mu \) is a Borel measure supported on \([-\pi/2, \pi/2]\), and the action of \( \nu \) on \( M \) is given by

\[
\langle f, \nu \rangle = \frac{1}{2\pi} \int_T f(x)\overline{g(x)} \, dx + \int_T f(x) \, d\mu(x), \quad f \in M.
\]

Clearly \( \|\nu\|_{M^*} \leq \max\{|\|g\|_2, |\mu|_T\} \) where \( |\mu|_T \) is the total variation of \( \mu \).

Suppose now that \( \nu = gdx + \mu \in M^* \) with \( g \) and \( \mu \) as described above, and assume that \( \sigma(\nu) \subseteq E_1 = 4\mathbb{Z} \). Then \( \hat{\nu}(n) = \int_T e^{-int} d\nu(t) = 0 \) for \( n \notin 4\mathbb{Z} \) and therefore for every trigonometric polynomial \( p \)

\[
\int_T p(t + \frac{\pi}{2}) \, d\nu(t) = \int_T p(t) \, d\nu(t),
\]

and since a continuous function on \( T \) is the uniform limit of trigonometric polynomials, the equality remains true if \( p \) is replaced by any such function. This shows that \( \nu \) is a measure of period \( \pi/2 \), that is, \( \nu(S + \pi/2) = \nu(S) \) for every Borel set \( S \subseteq T \). On the other hand, since \( \mu \) is supported on \([-\pi/2, \pi/2]\), \( \nu(S) = \int_S g(x) \, dx \) for every Borel set \( S \subseteq [0, \pi/2] \). Combining these facts we see that \( \nu \) is in \( L^2(T) \), and therefore setting \( S_N(\nu) = \sum_{n=-N}^{N} \hat{\nu}(n)e^{inx}, N = 0, 1, \ldots, \)

we deduce that

\[
\|\nu - S_N(\nu)\|_{M^*} \leq \|\nu - S_N(\nu)\|_2 \to 0 \quad \text{as} \quad N \to \infty.
\]

This shows that every element in \( M^* \) with spectrum contained in \( E_1 \), is even in the norm closure in \( M^* \), of the vector space spanned in \( M^* \) by the set \( \{e^{inx}: n \in E_1\} \), and therefore \( E_1 \) is of synthesis for \( M \). To show that \( E_2 = 4\mathbb{Z} + 2 \) is also a set of synthesis for \( M \), consider a measure \( \rho \) in \( M^* \) with spectrum in \( E_2 \); then the spectrum of the measure \( e^{2ix}\rho \) is in \( E_1 \), hence by the previous part of the proof this measure is in \( L^2(T) \), and therefore \( \rho \) is also in \( L^2(T) \). This implies as before, that \( E_2 \) is of synthesis for \( M \) (even in the norm topology of \( M^* \)). The remaining assertions of the theorem follow from the main result in [3]. For the sake of completeness we include here a different proof which is more in line with the approach of this paper. To show that the set \( E = E_1 \cup E_2 \) is not of synthesis for \( M \), consider the element of \( M^* \) defined by the measure \( \mu = \delta_{\pi/2} + \delta_{-\pi/2} \) (for every \( \tau \in T \), \( \delta_\tau \) denotes the unit point measure concentrated at \( \tau \)) and the function \( f \) in \( M \) defined by: \( f(x) = 1 \) for \(|x| \leq \pi/2 \) and \( f(x) = -1 \) for \( \pi/2 < |x| \leq \pi \). Then \( \hat{\mu}(n) = 2 \cos(n\pi/2), \forall n \in \mathbb{Z}, \hat{f}(n) = 2 \sin(n\pi/2)/n\pi, \) for \( n \in \mathbb{Z} \setminus \{0\}, \) and \( \hat{f}(0) = 0 \). Thus \( \sigma(\mu) \subseteq E \) and \( \hat{f} = 0 \) on \( E \), but \( \langle f, \mu \rangle = 2 \), and therefore \( E \) is not a set of synthesis for \( M \).
PROOF OF (b). To show that \( \{\infty\} \) is not a Ditkin set for \( M_1 \), consider again the measure \( \mu \) and the function \( f \) defined above. Noticing that \( \langle q, \hat{\mu} \rangle = 0 \) for every trigonometric polynomial \( q \) such that \( \hat{q} = 0 \) on \( 2\mathbb{Z} \), we obtain for every trigonometric polynomial \( p \) (by using the identity \( p \ast f = \sum_{n=-N}^{N} \hat{p}(n) \hat{f}(n)e^{inx} \), where \( N \) is the degree of \( p \)) that
\[
\|f - p \ast f\|_M \geq \|\mu\|_M^{-1} \langle f - p \ast f, \mu \rangle = 1
\]
and consequently \( \{\infty\} \) is not a Ditkin set for \( M_1 \).

REMARK. The proof of the theorem shows that the sets \( K_1 \) and \( K_2 \) are of synthesis for \( M_1 \) even in the norm topology of \( M_1^* \) but \( K_1 \cup K_2 \) is not of synthesis for \( M_1 \) even in the \( w^* \) topology of \( M_1 \).

The answers to the first and second questions mentioned at the beginning are not known for group algebras of locally compact noncompact abelian groups; in particular they are not known for the group algebra of \( \mathbb{Z} \). A discussion of these problems and partial results can be found in [1, Chapter 1 and 2], [4, Chapters 2 and 6] and [5, Chapter 7].

REFERENCES