that Sloman regards AI as a new synthesis of science and philosophy: *experimental philosophy*. He offers a challenge to those who bitterly attack AI:

> Anyone who objects to a particular explanation in the form of a program, should try to construct another better explanation of possibilities, that is, better according to the criteria by which explanations are assessed.... The preferred explanation should account for at least the same range of possibilities with at least as much fine structure. (p. 111)

Without insisting that it be a program, he does demand an equally complete explanation from any rival theory. Those who criticize AI should ponder this well.

If Sloman's book has the impact he hopes, it will certainly create what its title proclaims: a computer revolution in philosophy.

I have a few gripes with the way the book as a whole is put together: (1) it is riddled with typos and bad punctuation which do not impair understanding but which lower one's estimate for the amount the author cares about his work; (2) occasionally long passages appear in boldface, or reduced, or indented, for no apparent reason; (3) too many brief asides are thrown in for some special restricted audience, and they detract from the flow; (4) its tone is simply too biting.

But despite all my reservations, Sloman's book is a significant and highly original contribution to the debate about minds and machines.

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Interpolation theory, function spaces, differential operators, by Hans Triebel, North-Holland Publishing Company, Amsterdam, New York, Oxford, 1978, 528 pp., $66.75.

The first interpolation theorem was given by M. Riesz (1926) in his study of the L_p mapping properties of certain operators associated with the Fourier series. Riesz showed that the boundedness of a linear operator A as a mapping from $L_{p_i}(\mathbf{T})$ to $L_{r_i}(\mathbf{T})$ ($i = 0, 1$) carries with it the boundedness of A from $L_p(\mathbf{T})$ to $L_r(\mathbf{T})$ for other pairs (p, r). The power of the method is that it determines the mapping properties of A on L_p spaces by examining A on only two appropriate pairs of (endpoint) spaces.

Much of the early work in interpolation centered around extending and refining Riesz's results to be applicable to a larger variety of operators. It was not until the development of the abstract methods of interpolation in the late 1950s that the wide applicability of interpolation became clear. These abstract methods not only allow for the study of operators on general Banach spaces but also give a unified approach to the development of various classical families of spaces which arise in the modern theory of differential equations, approximation, and numerical analysis. With the development of these abstract methods, interpolation has become a major discipline which is indispensible for a thorough understanding of that portion of analysis which deals with spaces of functions and mappings of operators.

L_p theory. The early results of Riesz were refined by G. Thorin into what is now known as the Riesz-Thorin convexity theorem. If (S, μ) and (T, ν) are two nonnegative totally σ-finite measure spaces and A is a linear operator which maps $L_{p_i}(S)$ into $L_{r_i}(T)$ with norm M_i, $i = 0, 1$, then, A maps $L_p(S)$ into $L_r(T)$ with norm $M_\theta \leqslant M_0^{1-\theta}M_1^\theta$ for any pair (p, r), $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$, $0 \leqslant \theta \leqslant 1$. The admissible pairs (p, r) have a simple geometrical description. If we identify (p, r) with the point $(1/p, 1/r)$ in the unit square, then the admissible pairs (p, r) are exactly those for which $(1/p, 1/r)$ is on the line segment connecting $(1/p_0, 1/r_0)$ to $(1/p_1, 1/r_1)$. The set of such pairs is denoted by $\sigma[p_0, r_0; p_1, r_1]$ with closed or open brackets chosen to indicate whether or not the end pair is included.

Riesz applied this interpolation theorem to give a simple proof of the Hausdorff-Young estimates for Fourier series. If $f \in L_1(\mathrm{T})$ then a simple inequality gives that $\hat{f} \in l_\infty(Z)$ and $\|\hat{f}\|_\infty \leqslant \|f\|_1$. On the other hand Parseval's identity states that for any $f \in L_2(\mathrm{T})$, $\|\hat{f}\|_2 = \|f\|_2$. Thus the Riesz-Thorin convexity theorem applies and shows that $\|\hat{f}\|_r \leqslant \|f\|_p$ for any $(p, r) \in \sigma[1, \infty; 2, 2]$. The admissible pairs are those (p, r) with $1 \leqslant p \leqslant 2$ and $1/p + 1/r = 1$.

Weak L_p interpolation. Many interesting operators fail to be bounded on the appropriate endpoint spaces thereby precluding the direct application of the Riesz-Thorin theorem. The best known example is the Hilbert transform H which maps $L_p(R)$ into itself for $1 < p < \infty$ but not when $p = 1$ or ∞. To study such operators, a weak interpolation theory has been developed which replaces the assumption that A is bounded on the endpoint spaces by certain weaker, measure theoretic conditions. The ideas go back to J. Marcinkiewicz who introduced the condition

$$\nu\{x: |Af(x)| > y\} \leqslant cy^{-r}\|f\|_p^r \tag{1}$$

as a replacement for $A: L_p \to L_r$. The inequality (1) is automatically satisfied when A maps $L_p(S)$ into $L_r(T)$ since $y^r\nu\{x: |Af(x)| > y\} \leqslant \int_T |Af|^r\, d\nu$. Marcinkiewicz showed that if (1) holds for (p_i, r_i), $i = 0, 1$ then A maps $L_p(S)$ into $L_r(T)$ for all $(p, r) \in \sigma(p_0, r_0; p_1, r_1)$ provided $1 \leqslant p_1, p_2 \leqslant \infty$, $1 \leqslant r_1, r_2 < \infty$ and $p_i \leqslant r_i$, $i = 0, 1$ (the pairs are in the lower triangle).

The weak inequality (1) can be reformulated in terms of rearrangements of functions. This is an important step since it comes back to itegral inequalities between Af and f. For any measurable function f, the distribution function f is defined by $\lambda_f(y) \equiv \mu\{x: |f(x)| > y\}$ and the rearrangement of f is defined by $f^*(t) = \inf\{y: \lambda_f(y) \leqslant t\}$. The function f^* is defined on $(0, \infty)$ is nonincreasing and most importantly is equimeasurable with f and as such contains all the information about f needed for the L_p norms, indeed, $\|f^*\|_p = \|f\|_p$, $0 < p \leqslant \infty$. Rewriting (1) in terms of rearrangements gives

$$t^{1/r}(Af)^*(t) \leqslant C\|f^*\|_p, \qquad t > 0. \tag{2}$$

The formulation (2) leads naturally to the Lorentz spaces $L_{p,q}$ defined as the set of those functions f for which $\|f\|_{L_{p,q}} \equiv \|f^*\|_{p,q} < \infty$ where for any nonnegative nonincreasing function ψ,

$$\|\psi\|_{p,q} \equiv \left(\int_0^\infty \left[t^{1/p} \psi(t) \right]^q \frac{dt}{t} \right)^{1/q} \tag{3}$$

with the usual change to a sup when $q = \infty$. Certain fundamental embeddings hold for the Lorentz spaces, among others $L_{p,q_1} \subset L_{p,p} = L_p \subset L_{p,q_2}$ if $1 \leqslant q_1 \leqslant p \leqslant q_2 \leqslant \infty$. In terms of Lorentz spaces, (2) says that A maps L_p into $L_{r,\infty}$ so the weakening of the hypotheses in Marcinkiewicz's theorem consists of replacing L_r by the larger space $L_{r,\infty}$.

This idea can be carried even further in that L_p can be replaced by the smaller space $L_{p,1}$, namely, if A maps $L_{p_i,1}$ into $L_{r_i,\infty}, p_i \leqslant r_i, p_i \neq \infty$ $(i = 0, 1)$, then the conclusions of Marcinkiewicz's theorem hold. This follows (when $p > 1$) from the E. Stein-G. Weiss theorem **[13]** which shows that it is enough to consider simple functions in establishing the weak type of an operator. A. P. Calderón **[4]** made perhaps the most important step when he proved that an operator A maps $L_{p_i,1}$ into $L_{r_i,\infty}$, $i = 0, 1$, with $1 \leqslant p_0 < p_1 < \infty$ if and only if the following integral inequality holds.

$$(Af)^*(t) \leqslant \text{const } S_\sigma(f^*)(t), \qquad t > 0 \tag{4}$$

where for any decreasing ψ and any $\sigma = \sigma(p_0, r_0; p_1, r_1)$

$$\begin{aligned} S_\sigma(\psi) &\equiv \int_0^\infty \psi(s) \min_{i=0,1} \{ s^{1/p_i} t^{-1/r_i} \} \frac{ds}{s} \\ &= t^{-1/r_1} \int_0^{t^m} \psi(s) s^{1/p_i} \frac{ds}{s} + t^{1/p_2} \int_{t^m}^\infty \psi(s) s^{1/p_2} \frac{ds}{s} \end{aligned} \tag{5}$$

with m the slope of the line segment connecting $(1/p_1, 1/r_1)$, $(1/p_2, 1/r_2)$. When σ is closed on either endpoint the corresponding integral in (5) is deleted. This corresponds to strong mapping at that endpoint.

The integral inequality (4) can be used as a definition of weak type σ, as was done by C. Bennett **[1]**. It is preferable to (1) or any statement about mappings of spaces. For one thing, in applications, it is usually the inequality (4) which is derived when one studies the mapping properties of the operator A. Also, (4) makes it transparent that A inherits all the mapping properties of S_σ relative to rearrangement invariant norms. In fact, starting with (4), it is rather easy to prove the corresponding Marcinkiewicz theorems as well as other mapping results by merely applying norms to the inequality (4) and studying the mapping properties of S_σ. For Lorentz space norms the mapping properties of S_σ follow from the classical Hardy inequalities. There is another important reason for preferring (4). Our discussion so far has been restricted to the case $p_1 < \infty$. The reason for this is that the space $L_{\infty,1}$ according to (3) consists of only the zero function and therefore cannot be used in the definition of weak type when $p = \infty$. The usual remedy is to use L_∞ in place of $L_{\infty,1}$ but this returns us to (1) and in fact gives nothing new in the case (∞, ∞) since then weak type is the same as strong type $(L_{\infty,\infty} = L_\infty)$. On the other hand, (4) makes perfectly good sense when the parameters are infinite. We should further note that in the case $p_1 = \infty$, inequality (4) cannot be split into separate statements about mappings of Lorentz spaces.

Many operators satisfy weak type inequalities like (4). For example as was shown by Bennett and Rudnick [2], the R. O'Neil-G. Weiss inequality for the Hilbert transform can be sharpened to

$$(Hf)^*(t) \leqslant c\left(t^{-1}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)\frac{ds}{s}\right), \qquad 0 < t < \infty.$$

Thus H is of weak type $(1, 1; \infty, \infty)$. The standard mapping properties of H follow directly from this inequality e.g. H: $L_p \to L_p$, $1 < p < \infty$, H: $L \log L \to L_1$ (locally) and H: $L_\infty \to L_{\exp}$. Similar weak type inequalities hold for fractional integrals and singular integrals.

Abstract methods. Abstract methods were developed in the late 1950s and early 1960s which allow interpolation to be applied in a broad setting. These methods take two Banach spaces X_0, X_1 (which are assumed to be continuously embedded in a Hausdorff topological space) and use them to generate new spaces X_α, $0 < \alpha < 1$ which have the interpolation property, i.e. if A boundedly maps X_i into Y_i, $i = 0, 1$ then A boundedly maps X_α into Y_α, $0 \leqslant \alpha \leqslant 1$. There are in essence two approaches: the complex method which is patterned after the Riesz-Thorin theorem; and the real method which is similar to weak type interpolation for L_p spaces. The complex method was developed by A. P. Calderón, J. Lions, and S. G. Krein. Let $\mathfrak{F}$ denote the class of those functions f which take values in $X_0 + X_1$, are bounded, continuous on $0 \leqslant \operatorname{Re}(z) \leqslant 1$ and analytic in its interior, with $f(iy + t) \in X_t$, $t = 0, 1$. The space X_α is the set of all those $x \in X_0 + X_1$ for which $f(\alpha) = x$ for some $f \in \mathfrak{F}$. The norm on X_α is defined by $\|x\|_{X_\alpha} = \inf\{\|f\|;\ f \in \mathfrak{F}, f(\alpha) = x\}$ with

$$\|f\| = \max\Big(\sup_y \|f(iy)\|_{X_0}, \sup_y \|f(1 + iy)\|_{X_1}\Big).$$

If T is a linear operator mapping X_i into Y_i with norm M_i, $i = 0, 1$, then a simple application of the Hadamard three lines theorem shows that T maps X_α into Y_α with norm $M_\alpha \leqslant M_0^{1-\alpha}M_1^\alpha$ for each $0 < \alpha < 1$. Actually, the complex method can also be used to describe the mappings of a family of operators defined on a family of spaces depending on some analytic parameter and in fact this may be its most appropriate formulation [5], [12].

There are several real methods introduced by J. Peetre, J. Lions, E. T. Oklander and others. These various methods are equivalent at least in the sense that they generate the same interpolation spaces. The choice of the K method of Peetre is the most applicable. If $f \in X_0 + X_1$ and $t > 0$, then

$$\begin{aligned} K(f, t) &\equiv K(f, t, X_0, X_1) \\ &\equiv \inf\big\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_i \in X_i, i = 0, 1\big\}. \end{aligned}$$

In some cases, $X_1 \subseteq X_0$ and then $\|\cdot\|_{X_1}$ is only required to be a seminorm. For any f, $K(f, \cdot)$ is a nondecreasing continuous, concave function on $(0, \infty)$ and as such, $K(f, t)/t$ is nonincreasing. Thus, $K(f, \cdot)/(\cdot)$ has the same properties as f^* and now it is a simple matter to apply the Calderón-Marcinkiewicz ideas to this more general setting. For any $1 \leqslant p, q \leqslant \infty$, the space $X_{p,q}$ is the set of those $f \in X_0 + X_1$ for which $\|K(f, \cdot)/(\cdot)\|_{p,q} < \infty$ (see (3)).

A more standard notation is $X_{\theta,q}$ for $X_{p,q}$ where $\theta = 1 - 1/p$. When A is a linear operator which maps X_i into Y_i with norm M_i, $i = 0, 1$, then

$$K(Af, t, Y_0, Y_1) \leqslant M_0 K(f, M_1 t/M_0, X_0, X_1)$$

so that applying norms shows that A maps $X_{p,q}$ into $Y_{p,q}$ with norm $M_{p,q} \leqslant M_0^{1-\theta} M_1^{\theta}$ when $1 \leqslant p \leqslant \infty$ and $\theta = 1 - 1/p$.

The spaces $X_{p,q}$ can be thought of as abstract versions of the Lorentz spaces. Many families of spaces can be viewed either as X_α or $X_{p,q}$ spaces for appropriate endpoint spaces X_0, X_1. It is in this sense that interpolation theory can be used as a unified approach for the study of such spaces.

If A is a linear operator for which

$$K(Af, t, Y_0, Y_1)/t \leqslant \text{const } S_\sigma(K(f, \cdot, X_0, X_1)/\cdot)$$

then A is said to be generalized weak type σ with respect to (X_0, X_1), (Y_0, Y_1) [**6**]. When A is of weak type σ then for any $(p, r) \in \sigma$, A boundedly maps $X_{p,q}$ into $Y_{r,q}$, $1 \leqslant q \leqslant \infty$.

***K*-functionals.** What emerges as the key ingredient of the abstract real method is the K-functional. This is in essence what needs to be calculated in order to identify the spaces $X_{p,q}$. This has been done for several, but by no means all, classical pairs of endpoint spaces. In each such instance, the K-functional turns out to be an important analytic quantity which describes a family of classical spaces.

It is not surprising that the K-functional for (L_1, L_∞) is related to rearrangements of functions. J. Peetre and, independently, E. T. Oklander showed that

$$K(f, t, L_1, L_\infty) = \int_0^t f^*(s)\, ds \equiv tf^{**}(t), \qquad t > 0. \tag{6}$$

This means that $K(f, t, L_1, L_\infty)/t = S_\sigma(f^*)(t)$, for $\sigma = \sigma(1, 1; \infty, \infty]$. From the mapping properties of S_σ it follows that $X_{p,q} = L_{p,q}$ $(1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant \infty)$ for the pair (L_1, L_∞), and so at least in this sense the abstract real method recovers the classical L_p theory. There is a result of T. Holmstedt [**10**] which characterizes the K-functional for $(X_{p_0,q_0}, X_{p_1,q_1})$ in terms of the K-functional for (X_0, X_1). This can be combined with (6) to characterize the K-functional for any pair of Lorentz spaces.

There is an analogous result to (6) for the Sobolev spaces $W_p^k(\Omega)$ provided that $\Omega \subseteq R^n$ is suitably smooth [**7**],

$$K(f, t, W_1^k, W_\infty^k) \sim t \sum_{|\alpha|=k} (D^\alpha f)^{**}(t).$$

The spaces $X_{p,q}$ in this case are what could be called Lorentz-Sobolev spaces, in particular $X_{pp} = W_p^k$.

Another important case is interpolation between L_r and W_r^k, $1 \leqslant r \leqslant \infty$, $k = 1, 2, \ldots$. Here, H. Johnen and K. Scherer [**11**] have shown that for suitable smooth $\Omega \subseteq R^n$,

$$K(f, t, L_r, W_r^k) \sim \omega_k(f, t^{1/k})_r, \qquad t > 0 \tag{7}$$

with $\omega_k(f, \cdot)_r$ the kth order modulus of smoothness in L_r. The spaces X_{pq} are the Besov spaces $B_r^{\theta,q}$ wth $\theta = k - k/p$ and the norm on $B_r^{\theta,q}$ is

$$\|f\|_{B_r^{\theta,q}} \equiv \|f\|_r + \left(\int_0^\infty [t^{-\theta}\omega_k(f, t)_r]^q \frac{dt}{t}\right)^{1/q}.$$

There are two important K-functionals for harmonic analysis which have recently been found. C. Fefferman, N. Riviere, and Y. Sagher [8] have shown that

$$K(f, t; H_p, L_\infty) \sim \left(\int_0^{t^p} [(mf)^*(s)]^p \, ds\right)^{1/p}$$

where mf is the grand maximal function of f. For interpolation between L_1 and BMO we have the result of C. Bennett and R. Sharpley [3]

$$K(f, t, L_1, BMO) \sim t(f\#)^*(t)$$

where f is the "sharp function" introduced by C. Fefferman and E. Stein [9].

Properties of spaces. It is clear how interpolation gives information about the mapping properties of operators but it can also be used to give important information about the spaces themselves, in the form of embedding theorems, trace theorems and the like. This is often accomplished by studying the identity operator. For example, the inequality (for $\Omega \subseteq R^n$)

$$\omega_k(f, t)_{r_2} \leqslant c\int_0^t s^{-\theta}\omega_k(f, s)_{r_1} \frac{ds}{s} \tag{8}$$

when $\theta = n/r_1 - n/r_2$ expresses the fact that the identity operator is of weak type $\sigma(k/(k - \theta), 1; \infty, k/\theta]$ for the pairs $(L_{r_1}, W_{r_1}^k)$, $(L_{r_2}, W_{r_2}^k)$, because of (7). The standard embedding theorems for Besov and Sobolev spaces follow from (8), e.g. $B_{r_1}^{\lambda+\theta,q} \subseteq B_{r_2}^{\lambda,q}$, $\lambda > 0$. Embedding theorems for Besov spaces into Lorentz spaces can be proved from the inequality [6]

$$f^{**}(t) \leqslant c\left[\|f\|_r + \int_{t^{1/n}}^\infty s^{-n/r}\omega_n(f, s)_r \frac{ds}{s}\right] \tag{9}$$

for $r > 1$ (a strong inequality holds for $r = 1$).

Other information can be obtained from the study of differential operators. For example the weak inequalities

$$\omega_k(D^\beta f, t)_p \leqslant c\int_0^t \omega_{k+j}(f, s)_p \frac{ds}{s^{j+1}}, \qquad t > 0, |\beta| = j$$

establish the equivalence of norms involving (7) with other norms involving the modulus of smoothness of derivatives of f.

The book. The main thrust of Triebel's book is the development of Sobolev and Besov spaces from the viewpoint of interpolation theory. The first chapter gives a careful accounting of interpolation theory, pre-1975. Because of the continual activity in this subject, some of the more recent results, including the calculation of several K-functionals, is not included.

Chapters 2, 3, and 4 apply interpolation to the study of Sobolev and Besov spaces (with weights) including a unified development of embedding theorems, trace theorems and the like. There is much of interest here although in some instances the exposition suffers from poor orientation. A good case in point is the development of Besov spaces. The space $B_p^{\theta,q}$ is first defined for R^n using in essence the norm: $\|f\| = \|\{2^{\theta j}\|(\phi_j\hat{f})^{\vee}\|_{L_p}\}_{j=1}^{\infty}\|_{l_q}$, where $\wedge$ and $\vee$ the Fourier and inverse Fourier transforms and (ϕ_j) is a partition of unity with supp $\phi_j \subseteq \{x: 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}$. With some work, the equivalence of this norm with the norm involving (7) can be established. For more general $\Omega \subset R^n$, the space $B_p^{\theta,q}$ is defined as the set of those functions which can be extended to a function in $B_p^{\theta,q}(R^n)$. This approach not only loses the flavor of $B_p^{\theta,q}$ as a space of smooth functions, but all properties of Besov spaces must be verified through the cumbersome use of extensions and transforms. Most serious of all, this approach led to the decision of the author to exclude completely the cases $p = 1, \infty$, since these must be handled separately, in that the Hörmander multiplier theorem does not hold for these values of p.

Later chapters of the book deal with some selected topics including a study of (elliptic) differential operators and the structure of nuclear spaces. These establish once again the far reaching application of interpolation theory.

Triebel's book should surface as an important reference for interpolation theory and spaces of functions. The comprehensive notes and remarks section are of value even to the specialist.

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