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Approximants de Padé, by Jacek Gilewicz, Lecture Notes in Math., vol. 667, Springer-Verlag, Berlin-Heidelberg-New York, 1978, xiv + 511 pp., $21.50.

A rational function whose development in a Taylor series at the origin agrees term-by-term with a given formal power series C to a certain number of terms is called a Padé approximant of C. The recent literature on the subject contains a number of different formulations of the concept of Padé approximant (or fraction). Perhaps the most notable feature of Gilewicz' book is his definition of Padé approximant. Therefore I will try to describe this briefly in the following paragraph.

Let $K[Z]$ denote the algebra of polynomials in a variable z with coefficients in a field K (generally $K = \mathbf{C}$, the complex numbers). Let K^* and $K^*[Z]$ denote the non-null elements of K and $K[Z]$, respectively. Then in $K[Z] \times K^*[Z]$ an equivalence relation ω is defined by

$$(P, Q) \overset{\omega}{\sim} (P', Q') \Leftrightarrow \exists a \in K^*: P' = aP, Q' = aQ,$$
$$(P \in K[Z], Q \in K^*[Z]).$$

The equivalence class of an ordered pair (P, Q) modulo ω is called a *rational form* and is denoted by the symbol $P//Q$. The set quotient $K[Z] \times K^*[Z]/\omega$ of rational forms is denoted by $\mathcal{K}[Z]$. A second equivalence relation Ω in $K[Z] \times K^*[Z]$ is defined by

$$(P, Q) \overset{\Omega}{\sim} (P', Q') \Leftrightarrow PQ' - QP' = 0.$$

The relations ω and Ω are compatible so that Ω/ω is an equivalence relation in $\mathcal{K}[Z]$

$$P//Q \overset{\Omega/\omega}{\sim} P'//Q' \Leftrightarrow PQ' - QP' = 0.$$

The class of all $P//Q$ modulo Ω is called a *rational fraction* and is denoted by P/Q. In the class P/Q there exists a unique *reduced rational form* $P_1//Q_1$ such that

$$\forall P//Q \in P/Q, \exists R \in K^*[Z]: (P, Q) = (RP_1, RQ_1).$$

Let C denote a formal power series

$$C(Z) = \sum_{n=0}^{\infty} c_n z^n, \tag{1}$$

and let (M, N) be a given pair of nonnegative integers. Then a rational form $U_M//V_N$, determined by the equivalence class of all $(U_M, V_N) \in K[Z] \times K^*[Z]$ such that

$$\operatorname{ord}(CV_N - U_M) \geqslant M + N + 1, \qquad (\deg U_M \leqslant M, \deg V_N \leqslant N),$$

is called a *Padé form*. Here $\operatorname{ord}(CV_N - U_M)$ denotes the degree of the term of lowest degree in the formal power series $CV_N - U_M$ (where $\operatorname{ord}(0) = \infty$). The reduced rational form $P_M//Q_N$ generated by a Padé form $U_M//V_N$ is called the *reduced Padé form* (or, simply, the *reduced form*) and is denoted by r_{MN}. For the given (M, N), the set of all Padé forms $U_M//V_N$ determines a unique rational fraction P_M/Q_N and a unique reduced form $P_M//Q_N$. If the reduced form $P_M//Q_N$ is a Padé form, then the rational fraction P_M/Q_N is called a *Padé approximant*. If the reduced form $r_{MN} = P_M//Q_N$ is not a Padé form, then the Padé approximant is said to be *nonexistent*. Six different symbols are used to denote a Padé approximant of a formal power series C:

$$P_M/Q_N,\ P_{MN}/Q_{MN},\ p_{MN},\ [M/N]_C,\ [M/N] \text{ and } M/N.$$

Although most of these symbols have previously been used in the literature, the number seems to this reviewer to be excessive. The infinite array of reduced forms (r_{MN}) (all of which exist) is called the r-table, and the array of Padé approximants (p_{MN}) (which exist) is called the p-table of C. We note that in the literature some authors call the reduced form r_{MN} (or the unique rational function defined by r_{MN}) the (M, N) Padé approximant. In any case, it seems preferable to the reviewer to define a Padé approximant to be a rational function rather than a rational fraction, since one wishes to consider point-wise convergence of sequences of Padé approximants as well as differentiation and integration of them. Nevertheless, Gilewicz has introduced an interesting formulation of the concept of Padé approximants which is useful in clarifying some of the well-known subtleties of the subject. In the following we shall denote by p_{MN} both the rational fraction and the rational function (of the complex variable z) which it determines. If $C(z)$ is the Taylor expansion at $z = 0$ of an analytic function $f(z)$, then we call p_{MN} the (M, N) Padé approximant of $f(z)$ and we speak of the Padé table of $f(z)$.

If a Padé approximant p_{MN} of a series (1) exists, then it can be shown that

$$\operatorname{ord}(p_{MN} - C(z)) \geqslant M + N + 1; \tag{2}$$

that is, the Taylor series at $z = 0$ of p_{MN} agrees with $C(z)$ term-by-term up to and including the term of degree $M + N$. In the case when $C(z)$ is the Taylor expansion of a function $f(z)$ holomorphic at the origin, then (2) implies that

$$p_{MN}^{(k)}(0) = f^{(k)}(0), \qquad k = 0, 1, 2, \ldots, M + N. \tag{3}$$

These conditions define a Hermite interpolation problem, a solution of which may not exist among the rational functions of type $[M, N]$

$$\frac{\alpha_0 + \alpha_1 z + \cdots + \alpha_M z^M}{\beta_0 + \beta_1 z + \cdots + \beta_N z^N}.$$

When such a solution does exist, then it is unique and is given by the Padé approximant p_{MN}. Thus Padé approximants can be thought of as rational solutions of Hermite interpolation problems.

One of the main reasons why the Padé table of a function $f(z)$ is of interest is that various sequences of Padé approximants may converge in larger regions than the Taylor series of $f(z)$, which may not converge at all. The first column of the Padé table defines a sequence $\{p_{M,0}\}_{M=0}^{\infty}$ in which $p_{M,0}$ is the nth partial sum of the Taylor series at $z = 0$; it converges in the largest circular disk centered at the origin in which $f(z)$ is holomorphic. Another example of convergence of sequences of Padé approximants is given by the following result of Montessus de Ballore. If $f(z)$ is meromorphic in a domain containing the origin whose poles $z_1, z_2, z_3, \ldots$ are simple and such that

$$0 < |z_1| < |z_2| < |z_3| < \ldots,$$

then the Padé approximants of the nth column form a sequence $\{p_{MN}\}_{M=0}^{\infty}$, which converges to $f(z)$ uniformly on every compact subset of the disk $|z| < |z_{N+1}|$ containing none of the poles $z_1, z_2, \ldots, z_N$. In Chapter 6, Gilewicz treats these and other more recent convergence criteria for Padé approximants, such as convergence in measure (Nuttall) and in capacity (Pommerenke).

Continued fractions provide one of the most natural and powerful means for studying Padé tables. An example of a class of continued fractions whose approximants always lie in the Padé table consists of the regular C-fractions (or RITZ fractions) of the form

$$a_0 + \cfrac{a_1 z}{1 + \cfrac{a_2 z}{1 + \cfrac{a_3 z}{1 + \ddots}}}, \qquad a_n \neq 0 \text{ for } n \geqslant 1, \tag{4}$$

where the a_n are complex constants and z is a complex variable. For convenience (4) is frequently denoted by the symbol

$$a_0 + \frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_2 z}{1} + \cdots$$

and its nth approximant $f_n(z)$ is denoted by

$$f_n(z) = a_0 + \frac{a_1 z}{1} + \frac{a_2 z}{1} + \cdots + \frac{a_n z}{1}.$$

Corresponding to each regular C-fraction (5) there exists a unique power series $C(z)$ of the form (1) such that, for each $n \geqslant 1$, the Taylor expansion of $f_n(z)$ about $z = 0$ agrees with $C(z)$ term-by-term up to and including the term $c_n z^n$. The sequence of approximants $\{f_n(z)\}$ consists of the entries

$$p_{0,0}, p_{1,0}, p_{1,1}, p_{2,1}, p_{2,2}, p_{3,2}, \ldots,$$

which form a "staircase" in the Padé table of $C(z)$. If the continued fraction (5) (or, equivalently the sequence $\{f_n(z)\}$) converges uniformly on compact subsets of a domain D containing the origin to a function $f(z)$, then $f(z)$ is holomorphic in D and $C(z)$ is the Taylor expansion of $f(z)$ at $z = 0$. Among the well-known convergence theorems for regular C-fractions is the following. If

$$\lim a_n = a \neq 0,$$

then (5) converges to a meromorphic function $f(z)$, which is holomorphic at $z = 0$, and the convergence is uniform on compact subsets of $\mathbf{C}$ which contain no poles of $f(z)$. This result can be extended to the case with $a \neq 0$. If $a_n > 0$, for all n, then (5) is called a Stieltjes fraction (or S-fraction) and it converges to a function holomorphic in the cut plane $R = [z\colon |\arg z| < \pi]$ if and only if at least one of the series

$$\sum \frac{a_1 a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}}, \qquad \sum \frac{a_2 a_4 \cdots a_{2n-2}}{a_3 a_5 \cdots a_{2n-1}}$$

diverges. A sufficient condition for divergence of one of these series is that, for some $M > 0$,

$$|a_n| \leqslant M, \qquad n = 1, 2, 3, \ldots .$$

A simple example of an analytic function represented by an S-fraction is the natural logarithm

$$\log(1+z) = \frac{z}{1} + \frac{\left(\frac{1^2}{1\cdot 2}\right)z}{1} + \frac{\left(\frac{1^2}{2\cdot 3}\right)z}{1} + \frac{\left(\frac{2^2}{3\cdot 4}\right)z}{1} + \frac{\left(\frac{2^2}{4\cdot 5}\right)z}{1} + \cdots,$$

where the continued fraction converges throughout the z-plane cut along the real axis from -1 to $-\infty$. It represents the single-valued branch of $f(z) = \log(1 + z)$ such that $f(0) = 0$. The preceding remarks illustrate some of the wealth of convergence theory of continued fractions that is applicable to Padé tables. Gilewicz discusses both the regular C-fractions and J-fractions in Chapter 4. His main interest with continued fractions, however, is their relation to moment problems, dealt with in Chapter 3.

Another reason why there is a great deal of current interest in Padé tables is due to the development of efficient computational procedures for Padé approximants and related problems. The epsilon algorithm of Wynn and other algorithms due to Baker, Longman and Pindor are described in Chapter 7. These provide means for calculating not only the Padé approximants but also the zeros and poles of these rational functions. Chapter 8 deals with the best Padé approximant in a finite set of approximants and Chapter 9 describes applications of Padé approximants in numerical analysis, such as accelerating convergence of infinite series.

Among the notable omissions in the book we shall mention the following: (1) Applications of Padé approximants in physics, chemistry and engineering (see, for example, the two books [**1**] and [**2**] and the proceedings of international conferences on the subject [**4**], [**5**], [**7**], [**9**]). (2) A large body of material on continued fractions whose approximants lie in the Padé table. Some of these are recent work on truncation error analysis, algorithms such as Rutishauser's *qd*-algorithm, the work of Gautschi on minimal solutions of three-term recurrence relations, and the representation of special functions of mathematical physics by continued fractions. Modern treatments of these topics can be found in [**6**] and [**8**]. (3) Various generalizations and extensions of the concepts of Padé approximant such as the Laurent- and Chebychev-Padé approximants of Gragg and others, the multiple point Padé tables, and Padé approximants with several complex variables.

In view of the tremendous number of new papers on Padé approximants and related problems [**3**], Gilewicz' book provides a much needed supplement to the literature, since it brings together systematically a great deal of the recent material. The book contains approximately 500 pages which appear to be reproduced directly from the typescript. The first third of the book provides a concise summary of material, called "functional support," to which the author refers in the remaining chapters on Padé approximants. Very few proofs are given for known classical results. Although it is questionable whether his new formulation of the concept of Padé approximants will be uniformly adopted, it seems certain that Gilewicz' treatment of the subject will be widely used and cited.

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Barrelledness in topological and ordered vector spaces, by T. Husain and S. M. Khaleelulla, Lecture Notes in Math., vol. 692, Springer-Verlag, Berlin-Heidelberg-New York, 1978, x + 258 pp.

This book collects a vast number of facts that are scattered over the literature. Its subject can be divided into two more or less independent parts. The first part, which takes up about two thirds of the book, is concerned with topological vector spaces exclusively, and the second part with ordered topological vector spaces. This review will be divided into two parts accordingly.

Part I. In the proof of the classical Banach-Steinhaus theorem and the closed graph theorem, the category argument is used to establish the following:

(*) If U is an absorbing, convex, circled subset of a Banach space E, then the closure $\bar{U}$ of U is a neighborhood of 0 in E. (For subsets A and B of a linear space, A is said to *absorb* B if there exists a positive number λ_0 such