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Gaussian random processes, by I. A. Ibragimov and Yu. A. Rozanov, Applications of Math., volume 9, Springer-Verlag, New York-Heidelberg-Berlin, 1978, x + 276 pp., \$24.80.

A Gaussian law (= probability measure) P on a finite-dimensional vector space V is of the form $dP(x) = \exp(-Q(x)) dx_j$, where Q is a quadratic polynomial and dx_j is Lebesgue measure on a linear variety (affine subspace) J . Such laws, also called *normal*, are staples of multivariate statistics ([1], [34], [43]), along with their relatives such as Wishart distributions.

Let $EX = \int X dP$, the mean of the (vector or scalar) X . In the rest of this review *Gaussian laws will all have mean 0* unless otherwise stated. If A, B, C and D are any four linear forms on V , then $E(ABCD) = E(AB)E(CD) + E(AC)E(BD) + E(AD)E(BC)$. So, $E(A^4) = 3E(A^2)^2$, the first of a sequence of identities which characterize Gaussian laws on \mathbf{R}^1 .

Given a probability space $(\Omega, \mathfrak{B}, \text{Pr})$ and any set T , a *Gaussian process* is any real function X on $T \times \Omega$ such that for each finite set $F \subset T$, $\{X(t, \cdot)\}_{t \in F}$ has a Gaussian law on \mathbf{R}^F . Let $X(t) \equiv X(t, \cdot)$.

If T is a Hilbert space H , the *isonormal* Gaussian process L maps H isometrically into an $L^2(\Omega, \text{Pr})$, with $EL(x, \cdot)L(y, \cdot) = (x, y)$, the inner product; this fixes the laws of L . For any Gaussian process X , there is a Y with the same laws and $Y(t, \omega) = L(g(t), \omega)$, where g maps T into some Hilbert space H . So L is *the* Gaussian process [13]; it clothes a pristine Hilbert space in full Gaussian attire.

Trajectories. Probabilists like to pick an ω and follow the wandering path, or sample function, $t \rightarrow X(t, \omega)$ ([3], [13], [20], [48]). The speed at which $\exp(-x^2/2)$ goes to 0 as $x \rightarrow \infty$ lets us make (almost) all paths continuous if $g(T)$ in H is compact enough. If $T = \mathbf{R}$, the process X is called *stationary* if all its laws are preserved by translations $t \rightarrow t + h$. For a stationary X

restricted to a finite interval T , Fernique ([19], [20]) proved that “compact enough” can be exactly measured by Kolmogorov’s metric entropy: if you need $N(\epsilon)$ points to get within ϵ of all points of $g(T)$, then convergence of $\int_0^1 (\log N(u))^{1/2} du$ characterizes path-continuity (and is *sufficient* also for nonstationary Gaussian processes [13]), provided g is continuous.

Sudakov [55] characterizes sample continuity in terms of a mixed volume of infinite-dimensional convex sets. For some other recent sample function results see, e.g., [11], [12], [48].

General parameters. As knowledge of $X(t)$ for real t becomes refined, attention turns toward multidimensional t (“random fields”) and to linear processes $X(f, \cdot)$ on spaces of test functions f (“generalized random fields”), where the connecting idea is $X(f, \omega) = \int X(t, \omega) f(t) dt$. For one class of these, let $EN(f)N(g)^{-1} = \int (\mathcal{F}f)(\mathcal{F}g)^{-1} d\mu$ where \mathcal{F} denotes Fourier transform and μ is a nonnegative tempered measure. If $d\mu(y) = dy/(m^2 + |y|^2)$ for some $m > 0$, N is called a *Nelson process*, studied in quantum field theory ([8], [21], [44], [45], [50]).

Since a Gaussian process X_t (with mean 0) is characterized by its covariance $EX_t X_s$, one can look for covariances preserved by groups of isometries of symmetric spaces [2].

Abstract Wiener spaces and reproducing kernels. The process L on a Hilbert space H is not of the form $L(h, \omega) = (h, M(\omega))$ with $M(\omega) \in H$ (L is not sample-continuous). But if we restrict h to a dense, but small enough Banach subspace, we can take $M(\omega) \in B$ for any large enough Banach space B which is the completion of H for a small enough norm. L. Gross named such norms *measurable*; the arrangement (H, B) is called an *abstract Wiener space*, and seems to provide the best available substitute for Lebesgue measure in doing analysis on infinite-dimensional spaces ([9], [10], [15], [22], [23], [24], [25], [37], [38], [40], [47]); notable is Gross’ logarithmic Sobolev inequality [25]. Conversely, given B , there is an H : if P is a Gaussian law on a Banach space B , then there is a natural bounded linear map j of the dual B' into the Hilbert space $J = L^2(B, P)$. The adjoint j^* takes J onto a subspace $H \subset B \subset B''$. This H is the reproducing kernel Hilbert space $RKHS(P)$. These notions extend to spaces of sections of a vector bundle [4].

The Banach norms and spaces are, of course, not H -unitarily invariant. But one can think of the Gaussian measure “on H ” as concentrated on an infinite-dimensional sphere (surface) of radius $\sqrt{\infty}$, equipped with a Laplacian, spherical harmonics, etc. [42].

Analysis of functionals. For an orthonormal basis $\{e_n\}$ of a Hilbert space H_1 , the $L(e_n)$ are independent, identically distributed standard Gaussian variables X_n . Let $H := L^2(\Omega, P)$ be the space of all complex-valued functions $f = f(X_1, X_2, \dots)$ with $E|f|^2 < \infty$ (equivalence classes of measurable functions, actually). Then H is a countable orthogonal direct sum $\bigoplus_{n=0}^{\infty} H_{(n)}$, where $H_{(n)} = K_{(n)} \ominus \bigoplus_{j=0}^{n-1} H_{(j)}$ and $K_{(j)}$ is the set of all j th degree (or less) polynomials in the $L(x)$, $x \in H$. Let \mathfrak{H}_n be the n -fold symmetric tensor product of H_1 with itself, spanned by elements

$$\text{sym}(x_1 \otimes \cdots \otimes x_n) := (n!)^{-1} \sum_{\pi \in S(n)} x_{\pi(1)} \otimes \cdots \otimes x_{\pi(n)},$$

where $S(n)$ is the symmetric group of all permutations of $\{1, \dots, n\}$. Let $h \rightarrow :h:_{(n)}$ denote the orthogonal projection of H onto $H_{(n)}$. Then for each n , there is a map L_n such that for all $x_1, \dots, x_n \in H$, $L_n(\text{sym}(x_1 \otimes \dots \otimes x_n)) = :L(x_1) \dots L(x_n):_{(n)}$. For some constant c_n , $c_n L_n$ is an isometry of \mathfrak{H}_n onto $H_{(n)}$. This structural theory, developed by Wiener [56], von Neumann, Kakutani [36], and Segal [51], is quintessentially Gaussian; for expositions and more recent work see Neveu [46, Chapter 7], Hida [31], [32], Linnik [39], Guichardet [26], and Gutmann [27].

For a bounded linear operator A from H_1 into itself and each n , $A \otimes \dots \otimes A$ (n factors) maps \mathfrak{H}_n into itself. If $\|A\| \leq 1$, then these operators, via the above isometries, define a contraction $\Gamma(A)$ from H into itself. Nelson [45] proved a sharp inequality: if $1 \leq p \leq r \leq \infty$ and $\|A\| \leq ((p-1)/(r-1))^{1/2}$, then $\Gamma(A)$ is a contraction from $L^p(\Omega, P)$ into $L^r(\Omega, P)$.

Inequalities. Slepian [54] proved that if $EX_i^2 = EY_i^2$ and $EX_i X_j \leq EY_i Y_j$ for all i, j then $\sup_i X_i$ is stochastically larger than $\sup_i Y_i$. Several inequalities relate Gaussian laws and convex sets ([6], [7], [49]). Pitt [49] proved $P(A \cap B) > P(A)P(B)$ for P Gaussian and A and B symmetric convex sets in \mathbf{R}^2 (for \mathbf{R}^n , it's a conjecture). Some inequalities follow from the logarithmic concavity of Gaussian densities (e.g. [7]); others, from rotational invariance (e.g. [16]).

Equivalence and singularity. Hájek [28] proved that two Gaussian laws P and Q on a vector space are either singular or equivalent (mutually absolutely continuous). Here P and Q are equivalent if and only if the “ J -divergence” $(E_P - E_Q)\log(dP/dQ)$ is finite; it is the supremum of its finite-dimensional analogues. Using our general representation of Gaussian processes, nondegenerate P and Q can be written as affine transformations of each other, say $dQ(x) = dP(Ax + m)$; Segal [51] showed for the isonormal process, and Feldman ([17], [18]) proved in general, that P and Q are equivalent if and only if $m \in J = RKHS(P)$, and $A = I + B$ where B restricted to J is a Hilbert-Schmidt operator into J , with -1 not in its spectrum. Then A is extended from J to the larger space by continuity. So to find the relations of infinite-dimensional Gaussian laws, it helps to be able to recognize Hilbert-Schmidt operators in specific Hilbert spaces. From $L^2(\mu)$ to $L^2(\nu)$ they are just given by $L^2(\mu \times \nu)$ integral kernels. For later work on singularity, equivalence, and Radon-Nikodym densities, see e.g. Shepp [53] and the book under review.

Prediction. A stationary Gaussian process $X(t, \cdot)$ gives a one-parameter unitary group $U_h: X(t) \rightarrow X(t+h)$, acting on the Hilbert space(s) of the process. There is then a finite measure μ on \mathbf{R} , called the spectral measure, such that there is a linear isometry of $L^2(\mathbf{R}, \mu)$ into $L^2(\Omega, P)$ taking $(x \rightarrow e^{itx})$ to $X(t)$. Prediction and filtering of such processes are concerned with the closed linear spans X_A of $\{X(t): t \in A\}$ for subsets A ; or equivalently, with spans of $\{e^{itx}: t \in A\}$ in $L^2(\mathbf{R}, \mu)$: a matter of harmonic analysis. Classical prediction theory takes $A = [-\infty, s]$. Dym and McKean [14] treat this and other cases.

The review. Ibragimov and Rozanov's book actually treats three topics on stationary Gaussian processes (cf. also [30]): 1) singularity and equivalence, and calculation of densities (Radon-Nikodym derivatives) in case of equivalence; 2) in prediction, to find spectral measures μ for which X is “regular” or

“completely nondeterministic” in the sense that $\bigcap_n X_{]-\infty, -n]} = \{0\}$, and to study “mixing rates” for such processes; 3) in statistics, to estimate the mean $f(t)$ of a process $X(t, \cdot) + f(t)$ (“filtering”, cf. [35]). The list of references at the end of the book contains 28 items, mostly standard textbooks in analysis; 23 papers are cited in footnotes scattered through the volume.

Bits. Electrical engineering and information theory have, since Wiener’s fruitful intermediation, been in contact with Gaussian processes; recently flourishing related work is surveyed in [5] (level crossings), [35] (filtering), and [57].

A goodly number of functional limit theorems give Gaussian processes as limits—but that’s another story.

Reviews and bibliography. So far, authors of books and surveys have not tried to encompass the whole subject. Neveu [46] gave what is still the largest Gaussian bibliography, as far as I know, with some 600 items. Jain [33] and Marcus [41] gave courses. Each annual index of Mathematical Reviews currently lists between 50 and 100 papers on Gaussian processes (60G15). Of the 57 references below, 20 are themselves surveys or monographs, many of which have extensive bibliographies.

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Foundations of mechanics, Second Edition, Revised and enlarged, by Ralph Abraham and Jerrold E. Marsden, The Benjamin/Cummings Publishing Company, Reading, Mass., 1978, xxii + 806 pp., \$39.50.

1. This excellent book is one of several superb books on mechanics which have appeared in the past decade, such as those of Souriau [10], Siegel-Moser [9], Arnold [2] and Thirring [13], indicating a revitalized interest in the venerable subject of classical mechanics. Actually, there have been at least three sources of revitalization in the past forty years. The first came from the solution of the “small divisor problem” in celestial mechanics. The breakthrough here was achieved by Siegel in a mathematical tour de force, and then a new powerful general principle was discovered by Kolmogorov and developed in the hands of Arnold and Moser into a major analytical tool. The second came from the study of geometric properties of mappings and flows, especially in their “generic” behavior. The guiding philosophy had come from the foundational work in differential topology of Whitney and Thom, and was developed by Smale, Anosov, Sinai and their schools. More recently, there has been an influx of new ideas coming from group theory, from the work of Kirilov and Kostant in representation theory, and of Souriau, in rethinking the physical and geometrical principles underlying mechanics. As the bulk of the material added in the second edition deals with this last topic, I will concentrate my attention on it.

Much, but not enough, has been written about the philosophical problems