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PROJECTIONS OF C^∞ AUTOMORPHIC FORMS

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The purpose of this paper is to exhibit an explicit formula which describes the projection operator from the space of C^∞ automorphic forms to the subspace of holomorphic cusp forms, and to apply it to the zeta functions of Rankin type.

Fix a number $k > 0$ such that $2k \in Z$. Let N be a positive integer such that $N \equiv 0 \bmod(4)$ if $k \notin Z$, and let χ: $(Z/NZ) \longrightarrow \mathbf{C}$ be a Dirichlet character modulo N. Define

$$\Gamma_0(N) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(Z) | c \equiv 0 \bmod(N)\}$$

and $\mathfrak{H} = \{z = x + iy \in \mathbf{C} | y > 0\}$. For $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(Z)$ and $z \in \mathfrak{H}$, we put $\gamma(z) = (az + b)(cz + d)^{-1}$. For $b \geqslant 0$, denote by $\mathfrak{S}(k, N, \chi, b)$ the set of functions F satisfying

(1) F is a C^∞ function from $\mathfrak{H}$ to $\mathbf{C}$,

(2) $F(\gamma(z)) = \chi(d)j(k, \gamma, z)F(z)$ for all $\gamma \in \Gamma_0(N)$ where

$$j(k, \gamma, z) = \begin{cases} (cz + d)^k & \text{if } k \in Z, \\ \left(\dfrac{c}{d}\right)\left\{\left(\dfrac{-1}{d}\right)(cz + d)\right\}^k & \text{if } k \notin Z, \end{cases}$$

where (c/d) is the Legendre symbol (see Shimura [1] for a more complete explanation of this automorphy factor),

(3) $|F(z)| < C(y^a + y^{-b})$ for some positive real numbers C and a.

Let $G(k, N, \chi)$ be the set of all holomorphic modular forms satisfying condition (2) and let $S(k, N, \chi)$ be the subspace of $G(k, N, \chi)$ consisting of cusp forms.

Let $f \in S(k, N, \chi)$ and $F \in \mathfrak{S}(k, N, \chi, b)$. The Petersson inner product of f with F is defined as follows.

$$\langle f, F \rangle = m(N)^{-1} \int_{\Gamma_0(N)\backslash\mathfrak{H}} \overline{f(z)}F(z)y^{k-2}\, dxdy$$

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where $m(N)$ is the area of $\Gamma_0(N)\backslash\mathfrak{H}$ with respect to the measure $y^{-2}dxdy$. The functions F and f have Fourier expansions of the following type.

$$F(x+iy)=\sum_{n=-\infty}^{\infty} a(n,y)e(nx),$$

$$f(x+iy)=\sum_{n=1}^{\infty} a(n)e(nz),$$

where $e(x)=e^{2\pi ix}$. The functions $a(n,y)$ are C^∞ on $(0,\infty)$.

THEOREM 1. *Let $F\in\mathfrak{S}(k,N,\chi,b)$ with Fourier expansion as above. Assume that $k>2$ and $b<k-1$. Let*

$$c(n)=(2\pi n)^{k-1}\Gamma(k-1)^{-1}\int_0^\infty a(n,y)e^{-2\pi ny}y^{k-2}\,dy.$$

Then $h(z)=\Sigma_{n=1}^\infty c(n)e(nz)\in S(k,N,\chi)$. Moreover, $\langle g,F\rangle=\langle g,h\rangle$ for all $g\in S(k,N,\chi)$.

The function h is denoted by $h=P(F)$.

Theorem 1 can be used to study the Rankin zeta function.

For $f(z)=\Sigma_{n=1}^\infty a(n)e(nz)\in S(k,N,\chi)$ and $g(z)=\Sigma_{n=1} b(n)e(nz)\in G(r,N,\psi)$, define

$$D(s,f,g)=\sum_{n=1}\overline{a(n)}b(n)n^{-s}.$$

Then $D(s,f,g)$ can be analytically continued to a meromorphic function on the whole s-plane. There is a unique cusp form $K(k,g,s)\in S(k,N,\chi)$ such that

$$\langle f,K(k,g,s)\rangle=D(s,f,g)$$

for all $f\in S(k,N,\chi)$. The Fourier expansion of $K(k,g,s)$ will now be determined for s inside a vertical strip of the complex plane.

Let R be a set of representatives for $\Gamma_\infty\backslash\Gamma_0(N)$ where

$$\Gamma_\infty=\{\pm\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)\mid m\in Z\},$$

and put

$$E(z,s)=\sum_{\gamma\in R}\chi(d)\psi(d)j(k,\gamma,z)j(r,\gamma,z)|cz+d|^{-2(s+k)}.$$

Rankin's representation of $D(s,f,g)$ as an inner product can be stated as follows.

$$(4\pi)^{-s}\Gamma(s)D(s,f,g)=\langle f,gE(z,s+1-k)y^{s+1-k}\rangle m(N).$$

Hence $K(k,g,s)=P(gE(z,s+1-k)y^{s+1-k})(4\pi)^s\Gamma(s)^{-1}m(N)$. In order to obtain the Fourier expansion of $K(k,g,s)$ from Theorem 1, it is necessary to know the Fourier expansion of the function $E(z,s)$.

For $y > 0$ and $(\alpha, \beta) \in \mathbf{C}^2$, define

$$W(y, \alpha, \beta) = \Gamma(\beta)^{-1} \int_0^\infty (u+1)^{\alpha-1} u^{\beta-1} e^{-yu}\, du.$$

The integral is absolutely convergent for $\mathrm{Re}(\beta) > 0$, and $W(y, \alpha, \beta)$ can be analytically continued to a holomorphic function on all of $\mathbf{C}^2$. If $y > 0$ define $W(y, \alpha, \beta) = W(-y, \beta, \alpha)$.

The function $E(z, s)$ has a Fourier expansion of the following type

$$E(z, s) = c(y, s) + \sum_{t \neq 0} a(t, s) W(4\pi t y, k - r + s, s) e^{-2|t|\pi y} e(xt).$$

The functions $a(t, s)$ are known. When $k - r \in Z$, they are simple arithmetic functions, and when $k - r \notin Z$, they are, up to simple arithmetic factors, Dirichlet L-functions [1], [4].

Let $c \geqslant 0$ be such that $b(n) = O(n^{c+\epsilon})$ for every $\epsilon > 0$. Put

$$I(s, t, u) = \begin{cases} \int_0^\infty W(4\pi t y, s+1-r, s+1-k) e^{-2\pi(|t|+u)y} y^{s-1}\, dy & \text{if } t \neq 0 \\ \int_0^\infty c(y, s+1-k) e^{-2\pi u y} y^{s-1}\, dy & \text{if } t = 0. \end{cases}$$

THEOREM 2. *Let*

$$c(u, s) = (2\pi)^{k-1} \Gamma(k-1)^{-1} u^{k-1} \sum_{n+t=u} b(n) a(t, s+1-k) I(s, t, u) m(N),$$

where $a(0, s) = 1$. *Then this sum converges for* $c + 1 < \mathrm{Re}(s) < k + r - 2 - c$, *and for such* s, *the function* $K(k, g, s) = \Sigma_{u=1}^\infty c(u, s) e(uz) \in S(k, N, \chi)$. *Furthermore,* $(4\pi)^{-s}\Gamma(s) D(s, f, g) = \langle f, K(k, g, s)\rangle$ *for all* $f \in S(k, N, \chi)$.

REMARK. In [5], Zagier (using a different method) computes $K(k, g, s)$ for $g(z) = \theta(z) = \Sigma_{n=-\infty}^\infty e(n^2 z)$, $k \in Z$ and $N = 1$. He shows that the nth Fourier coefficient of $K(k, g, k/2)$ is essentially the trace of the nth Hecke operator, and in this way, he recovers the Eichler-Selberg trace formula for $SL_2(Z)$. Using Theorem 2, it seems possible to recover the trace formula arbitrary congruence subgroups of $SL_2(Z)$.

Now assume that $k, r \in Z$ with $k > r > 0$. Then, for $m \in Z$, $r < m < k - 1$, the function $K(k, g, m)$ has a simpler form. In fact, for those special values of m, the integrals which appear in Theorem 2 can be evaluated and the sums defining $c(u, m)$ are finite.

In order to write down the Fourier expansion of $K(k, g, m)$ for $m \in J = \{m \in Z | r < m < k - 1\}$, it is necessary to introduce some notation. Let

$$D_N(s, f, g) = L(2s + 2 - k - r, \chi\psi) D(s, f, g).$$

If ω is a character modulo N, and $d \in Z$, define

$$\mathfrak{G}(\omega, d) = \sum_{q=1}^{N-1} \omega(q)e(dq/N) \quad \text{and} \quad \delta(\omega) = \begin{cases} 0 & \text{if } \omega \neq I_N, \\ 1 & \text{if } \omega = I_N, \end{cases}$$

where I_N is the trivial character modulo N. For $m \in J$, $t > 0$ and $u > 0$, define $I(m, t, u)$

$$= \pi^{-m} \sum_{i=0}^{m-r} \binom{m-r}{i} \left\{ \prod_{j=0}^{i} (m + j - k) \right\} t^{k-m-i-1} (t + u)^{k-i-1} 2^{k-i-2m-1},$$

$$a(t, m) = i^{r-k}(2\pi/N)^{2m+2-r-k}\Gamma(m + 1 - r)^{-1} \cdot \sum_{d/t} d^{2m+1-r-k}(\mathfrak{G}(\chi\psi, d) + V\mathfrak{G}(\chi\psi, -d)),$$

where $V = (-1)^{k-r}$. Let

$$\begin{aligned} \xi(s) = {} & 2L(2s + 2 - r - k, \chi\psi)(2\pi u)^{-s}\Gamma(s) \\ & + \{(2\pi)^{s+2-r-k} i^{r-k} N^{2s-r-k} \varphi(N)\delta(\omega)\Gamma(r + k + s - 1)\Gamma(2s + 1 - r - k) \\ & \cdot \Gamma(s + 1 - k)^{-1}\Gamma(s + 1 - r)^{-1}\} 2L(2s + 1 - r - k, I_N), \end{aligned}$$

where φ is the Euler φ-function. For $u > 0$ define

$$c(u, m) = (2\pi u)^{k-1}\Gamma(k - 1)^{-1} \sum_{t=1}^{u} b(u - t)a(t, m)I(m, t, u) + b(u)\xi(m) .$$

Let $K_N(k, g, m) = \sum_{u=1}^{\infty} c(u, m)e(uz)$.

COROLLARY. *If $r < m < k - 1$, then*

$$m(N)^{-1}(4\pi)^{-m}\Gamma(m)D_N(m, f, g) = \langle f, K_N(k, g, m)\rangle.$$

The Fourier coefficients of $K_N(k, g, m)$ are explicitly given by the above formulas. This corollary supplements Shimura's theorem (Theorem 4 of [3]).

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