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$K_r(\mathbf{Z}/p^2)$ AND $K_r(\mathbf{Z}/p[\epsilon])$ FOR $p \geqslant 5$ AND $r \leqslant 4$

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If R is a ring, $K_0(R)$ is the Grothendieck group of finitely generated projective R-modules, $K_1(R)$ is the abelianization of the group $GL(R)$ of invertible matrices over R, and $K_2(R)$ is the second homology group of $E(R) = \ker(GL(R) \longrightarrow K_1(R))$. Higher K-groups are defined as homotopy groups of a space associated to $GL(R)$ and provide additional homological invariants of the linear algebra of R. Unfortunately, these higher (degree greater than 2) K-groups appear difficult to compute even for very simple rings: in particular, no higher K-groups of rings with nilpotents have been computed. We present computations for two such rings, $\mathbf{Z}/p^2\mathbf{Z}$ and $\mathbf{Z}/p[\epsilon]$ (the dual numbers over $\mathbf{Z}/p$).

Before stating our results, we briefly mention other computations of higher K-groups. Quillen [9] computed $K_i(\mathbf{F}_q)$ for any $i \geqslant 0$ and any finite field $\mathbf{F}_q$. Browder [3], Harris and Segal [6], Quillen [11], and Soule [12] have partial results on higher K-groups of rings of integers in number fields. Borel [2] has computed the ranks of the K-groups of such rings. Lee and Szczarba [7] have computed $K_3(\mathbf{Z})$. Moreover, Quillen [10] has proved many general theorems which enable one to convert known computations of various rings to computations of related rings.

We announce the following theorems whose proofs will appear in [5].

THEOREM 1. *Let $p \geqslant 5$ be a prime. Let $\mathbf{Z}/p[\epsilon]$ denote the ring (of order p^2) of dual numbers over $\mathbf{Z}/p$.*

$$K_1(\mathbf{Z}/p^2) = K_1(\mathbf{Z}/p[\epsilon]) = \mathbf{Z}/p - 1 \oplus \mathbf{Z}/p,$$

$$K_2(\mathbf{Z}/p^2) = K_2(\mathbf{Z}/p[\epsilon]) = 0,$$

$$K_3(\mathbf{Z}/p^2) = \mathbf{Z}/p^2 - 1 \oplus \mathbf{Z}/p^2; K_3(\mathbf{Z}/p[\epsilon]) = \mathbf{Z}/p^2 - 1 \oplus \mathbf{Z}/p \oplus \mathbf{Z}/p,$$

$$K_4(\mathbf{Z}/p^2) = K_4(\mathbf{Z}/p[\epsilon]) = 0.$$

Of course, $K_1(\mathbf{Z}/p^2)$ and $K_1(\mathbf{Z}/p[\epsilon])$ are well known [1, V. 9.1], $K_2(\mathbf{Z}/p^2)$ was computed by Milnor [8], and $K_2(\mathbf{Z}/p[\epsilon])$ was computed by van der Kallen [13].

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Our proof of Theorem 1 is based on the following homology calculation.

THEOREM 2. *Let $p \geq 5$ be a prime and let $SL(p^2) = \varinjlim SL(n, \mathbf{Z}/p^2)$ and $SL(\epsilon) = \varinjlim SL(n, \mathbf{Z}/p[\epsilon])$.*

$$H_1(SL(p^2)) = H_1(SL(\epsilon)) = 0,$$

$$H_2(SL(p^2)) = H_2(SL(\epsilon)) = 0,$$

$$H_3(SL(p^2)) = \mathbf{Z}/p^2 - 1 \oplus \mathbf{Z}/p^2; H_3(SL(\epsilon)) = \mathbf{Z}/p^2 - 1 \oplus \mathbf{Z}/p \oplus \mathbf{Z}/p,$$

$$H_4(SL(p^2)) = H_4(SL(\epsilon)) = 0.$$

Let $R = \mathbf{Z}/p^2$ or $\mathbf{Z}/p[\epsilon]$. Then $K_1(R) = H_1(SL(R)) \oplus R^{\cdot}$, $K_i(R) = \pi_i(BSL(R)^+)$ for $i > 1$, and $H_i(SL(R)) = H_i(BSL(R)^+)$. Therefore, Theorem 2 and the Hurewicz Theorem imply the computations of $K_1(R)$, $K_2(R)$, and $K_3(R)$ of Theorem 1. Furthermore, $K_4(R)$ is obtained from Theorem 2 using the Serre spectral sequence for the natural map $BSL(R)^+ \longrightarrow K(K_3(R), 3)$ and the well-known values of the $\mathbf{Z}/p$ homology of $K(K_3(R), 3)$.

The proof of Theorem 2 is achieved by considering $SL(n, \mathbf{Z}/p^2) = SL(n, p^2)$ and $SL(n, \mathbf{Z}/p[\epsilon]) = SL(n, \epsilon)$ as extensions over $SL(n, \mathbf{Z}/p) = SL(n, p)$. Because Quillen determined $H_*(SL(p), \mathbf{Z})$ in [9] and because the kernels of $SL(n, p^2) \longrightarrow SL(n, p)$ and $SL(n, \epsilon) \longrightarrow SL(n, p)$ are p-groups, the content of Theorem 2 is its determination of the p-primary component of the asserted homology groups.

Let $H_*(G, A; p)$ denote the p-primary component of $H_*(G, A)$ for any group G and G-module A. We consider the spectral sequence

$$E^2_{i,j}(p^2, \mathbf{Z}) = H_i(GL(n, p), H_j(V_n); p) \Rightarrow H_{i+j}(\overline{SL}(n, p^2), \mathbf{Z}; p)$$

where $1 \longrightarrow V_n \longrightarrow \overline{SL}(n, p^2) \longrightarrow GL(n, p) \longrightarrow 1$ is the restriction of the extension $1 \longrightarrow M_n \longrightarrow GL(n, p^2) \longrightarrow GL(n, p) \longrightarrow 1$ to the subgroup $\overline{SL}(n, p^2)$ of $GL(n, p^2)$ consisting of matrices whose determinant has order prime to p. We also consider the analogous spectral sequence $\{E^r_{i,j}(\epsilon, \mathbf{Z})\}$ for $H_*(\overline{SL}(n, \epsilon), \mathbf{Z}; p)$; then $E^2_{i,j}(p^2, \mathbf{Z}) = E^2_{i,j}(\epsilon, \mathbf{Z})$. To prove Theorem 2, it suffices to compute $H_r(\overline{SL}(n, p^2), \mathbf{Z}; p)$ and $H_r(\overline{SL}(n, \epsilon), \mathbf{Z}; p)$ which is done using these spectral sequences. To identify $H_3(\overline{SL}(n, p^2), \mathbf{Z}; p)$ and $H_3(\overline{SL}(n, \epsilon), \mathbf{Z}; p)$ precisely and not simply their associated graded structures given by these spectral sequences, we also must consider $\{E^r_{i,j}(p^2, \mathbf{Z}/p)\}$ and $\{E^r_{i,j}(\epsilon, \mathbf{Z}/p)\}$ (which have isomorphic E_2-terms).

The analysis of these spectral sequences involves the determination of $E^2_{i,j}$ for $i + j \leq 4$ and the identification of all relevant differentials. For example,

$$E^2_{0,3}(\mathbf{Z}/p^2, \mathbf{Z}) = H_0(GL(n, p), \Lambda^3 V_n \oplus S^2 V_n) = \mathbf{Z}/p \oplus \mathbf{Z}/p,$$

$$E^2_{2,2}(\mathbf{Z}/p^2, \mathbf{Z}) = H_2(GL(n, p), \Lambda^2 V_n) = \mathbf{Z}/p.$$

The calculations of $E^2_{i,j}$ are made by computing the homology groups

$$H_i(B_n, H_j(V_n); p) \qquad (= H_i(B_n, H_j(V_n)) \text{ for } j > 0)$$

where B_n is the subgroup of $GL(n, \mathbf{Z}/p)$ of $n \times n$ upper triangular matrices. For $j > 0$, $H_j(V_n)$ is considered with a convenient filtration as a B_n module and the spectral sequence of this filtered module is employed:

$$E^1_{s,t} = H_{s+t}(B_n, F_sH_j(V_n)/F_{s-1}H_j(V_n)) \Rightarrow H_{s+t}(B_n, H_j(V_n)).$$

The necessary E^1-terms of this spectral sequence are computed using the projection map $B_n \longrightarrow B_{n-1}$ and induction; the necessary differentials are computed explicitly.

The only possible nonzero differentials in the spectral sequences $\{E^r_{i,j}(p^2, \mathbf{Z})\}$, $\{E^r_{i,j}(\epsilon, \mathbf{Z})\}$, $\{E^r_{i,j}(p^2, \mathbf{Z}/p)\}$, and $\{E^r_{i,j}(\epsilon, \mathbf{Z}/p)\}$ in the range under consideration are the differentials

$$d^2_{2,2}\colon E^2_{2,2} \longrightarrow E^2_{0,3}.$$

Because of the stability with respect to n of $E^2_{2,2}$ and $E^2_{0,3}$, it suffices to consider the case $n = 2$. For $d^2_{2,2}\colon E^2_{2,2}(\epsilon, \mathbf{Z}) \longrightarrow E^2_{0,3}(\epsilon, \mathbf{Z})$ and $d^2_{2,2}\colon E^2_{2,2}(\epsilon, \mathbf{Z}/p) \longrightarrow E^2_{0,3}(\epsilon, \mathbf{Z}/p)$, we employ an explicit cocycle calculation for the split extension

$$1 \longrightarrow V_2 \longrightarrow \overline{SL}(2, \epsilon) \underset{GL(2,p)}{\times} B_2 \longrightarrow B_2 \longrightarrow 1.$$

For $d^2_{2,2}\colon E^2_{2,2}(p^2, \mathbf{Z}) \longrightarrow E^2_{0,3}(p^2, \mathbf{Z})$ and $d^2_{2,2}\colon E^2_{2,2}(p^2, \mathbf{Z}/p) \longrightarrow E^2_{0,3}(p^2, \mathbf{Z}/p)$, we use the determination of $d^2_{2,2}$ in the split case together with the theory of Charlap and Vasquez [4] to identify these differentials.

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