

REALIZATIONS OF NONLINEAR SYSTEMS AND ABSTRACT TRANSITIVE LIE ALGEBRAS

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Introduction. We shall derive nonlinear analytic realizations in system theory from infinite transitive Lie algebras and Lie pseudogroups. By using noncommutative generating power series, this new approach allows a local input-output viewpoint which was not possible until now (cf. [5], [6], [9], [10]) and should lead, thanks to the notion of *syntactic* Lie algebra, to many developments.

I. Review of noncommutative generating power series (cf. [2], [3]). X^* is the free monoid generated by $X = \{x_0, x_1, \dots, x_n\}$ and 1 is its identity element. Let $\mathbf{R}\langle X \rangle$ and $\mathbf{R}\langle\langle X \rangle\rangle$ be the \mathbf{R} -algebras of formal polynomials and power series with real coefficients and associative variables $x_j \in X$ (noncommutative if $n \geq 1$).

A causal functional $F(t; u_1, \dots, u_n)$, where $u_1, \dots, u_n: [0, T] \rightarrow \mathbf{R}$ are piecewise continuous functions, is said to be *analytic* iff it is given by an element

$$g = (g, 1) + \sum_{\nu \geq 0} \sum_{j_0, \dots, j_\nu = 0}^n (g, x_{j_\nu} \dots x_{j_0}) x_{j_\nu} \dots x_{j_0}$$

of $\mathbf{R}\langle\langle X \rangle\rangle$, called its *generating* power series, such that its value is

$$(1) \quad F(t; u_i) = (g, 1) + \sum_{\nu \geq 0} \sum_{j_0, \dots, j_\nu = 0}^n (g, x_{j_\nu} \dots x_{j_0}) \int_0^t d\xi_{j_\nu} \dots d\xi_{j_0}.$$

The iterated integral is defined recursively on the length

$$\xi_0(\tau) = \tau, \quad \xi_i(\tau) = \int_0^\tau u_i(\sigma) d\sigma \quad (i = 1, \dots, n),$$

$$\int_0^\tau d\xi_j = \xi_j(\tau) \quad (j = 0, 1, \dots, n),$$

$$\int_0^t d\xi_{j_\nu} \dots d\xi_{j_0} = \int_0^t d\xi_{j_\nu}(\tau) \int_0^\tau d\xi_{j_{\nu-1}} \dots d\xi_{j_0}.$$

HYPOTHESIS (H). (1) is absolutely convergent for t and $\max_{0 \leq \tau \leq t} |u_i(\tau)|$ sufficiently small.

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Let us introduce the differential system

$$\begin{cases} \dot{q}(t) (= dq/dt) = A_0(q) + \sum_{i=1}^n u_i(t)A_i(q) \\ y(t) = h(q), \end{cases}$$

where $q(t)$ belongs to a real analytic manifold Q and $q(0)$ is given; A_0, A_1, \dots, A_n are analytic vector fields and $h: Q \rightarrow \mathbf{R}$ is an analytic function, which are defined in a neighbourhood of $q(0)$. The output y is an analytic functional with the following generating power series

$$g = h|_{q(0)} + \sum_{v \geq 0} \sum_{j_0, \dots, j_v=0}^n A_{j_0} \dots A_{j_v} h|_{q(0)} x_{j_v} \dots x_{j_0},$$

where the bar $|_{q(0)}$ indicates the evaluation at $q(0)$.

II. Results. Let $H(g)$ be the Hankel matrix (cf. [1]) of $g \in \mathbf{R}\langle\langle X \rangle\rangle$. The \mathbf{R} -vector space spanned by its columns has the canonical structure of a left $\mathbf{R}\langle X \rangle$ -module: the product $u \cdot c_v$ of $u \in X^*$ and the column with index $v \in X^*$ is the column $c_{uv} \in H(g)$.

$L(X)$ is the free Lie algebra the envelopping algebra of which is $\mathbf{R}\langle X \rangle$. The *Lie rank* of g is the dimension of the \mathbf{R} -vector space spanned by $\{pc_1 | p \in L(X)\}$.

THEOREM. (A) *For $g \in \mathbf{R}\langle\langle X \rangle\rangle$, the two following conditions are equivalent:*

- (i) *The Lie rank of g is finite, equal to N .*
- (ii) *There exist a commutative power series $h \in \mathbf{R}[[q^1, \dots, q^N]]$ and formal vector fields*

$$A_j = \sum_{k=1}^N \theta_j^k \frac{\partial}{\partial q^k} \quad (\theta_j^k \in \mathbf{R}[[q^1, \dots, q^N]]; j = 0, 1, \dots, n),$$

defined up to equivalence, such that

$$g = h|_0 + \sum_{v \geq 0} \sum_{j_0, \dots, j_v=0}^n A_{j_0} \dots A_{j_v} h|_0 x_{j_v} \dots x_{j_0},$$

where the bar $|_0$ indicates the evaluation at $q^1 = \dots = q^N = 0$.

(B) *When hypothesis (H) is satisfied, it is possible to choose the series θ_j^k and h to be convergent in a neighbourhood of 0. This choice is unique up to equivalence.*

III. Sketch of the proof. Let I be the ideal of $L(X)$ defined by

$$I = \{p \in L(X) | \forall k \geq 0, \forall p_1, \dots, p_k \in L(X), [p_k, \dots [p_1, p] \dots] c_1 = 0\}.$$

We shall say that $\mathfrak{U}(\mathfrak{g}) = L(X)/I$ is the *syntactic* Lie algebra of \mathfrak{g} which can be looked upon as an *abstract transitive* Lie algebra (cf. [4], [8]) with the following *fundamental* subalgebra:

$$L^0 = \{p \in \mathfrak{U}(\mathfrak{g}) \mid pc_1 = 0\}.$$

The Lie rank of \mathfrak{g} is equal to the dimension of $\mathfrak{U}(\mathfrak{g})/L^0$. As a result, part A of the theorem follows from [4, Theorem III], or [8, Theorem 4.3].

When hypothesis (H) is satisfied, the corresponding local Lie pseudogroup can be constructed with the same kind of methods used in [6, §5]. The uniqueness up to equivalence follows from [8, §3.2], or [7].

REFERENCES

1. M. Fliess, *Matrices de Hankel*, J. Math. Pures Appl. **53** (1974), 197–222.
2. ———, *Développements fonctionnels en indéterminées non commutative des solutions d'équations différentielles non linéaires forcées*, C. R. Acad. Sci. Paris A287 (1978), 1133–1135.
3. ———, *Fonctionnelles causales non linéaires et indéterminées non commutatives* (submitted).
4. V. R. Guillemin and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964), 16–47.
5. R. Hermann and A. J. Krener, *Nonlinear controllability and observability*, IEEE Trans. Automat. Control **22** (1977), 728–740.
6. B. Jakubczyk, *Existence and uniqueness of realizations of nonlinear systems*, Preprint 148, Instit. Math., Warsaw, 1978.
7. A. M. Rodrigues and A. Petitjean, *Correspondance entre algèbres de Lie abstraites et pseudo-groupes de Lie transitifs*, Ann. of Math. (2) **101** (1975), 268–279.
8. I. M. Singer and S. Sternberg, *The infinite groups of Lie and Cartan. I*, J. Analyse Math. **15** (1965), 1–114.
9. H. J. Sussmann, *A generalization of the closed subgroup theorem to quotients of arbitrary manifolds*, J. Differential Geometry **10** (1975), 151–166.
10. ———, *Existence and uniqueness of minimal realizations of nonlinear systems*, Math. Systems Theory **10** (1977), 263–284.

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