

THE NIELSEN REALIZATION PROBLEM

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If M_g is a closed, oriented 2-manifold of genus $g \geq 2$, then it admits many hyperbolic metrics (metrics of constant curvature -1). In special cases such a metric possesses a nontrivial group of symmetries, of isometries to itself. The group of isometries of a closed hyperbolic manifold is always finite and the only isometry isotopic to the identity is the identity itself. Thus a group of symmetries of a hyperbolic surface determines an isomorphic finite subgroup of the group of isotopy classes of diffeomorphisms of M_g . The purpose of this paper is to announce a positive solution to the *Nielsen Realization Problem* that the converse is true, i.e.; the

THEOREM 1. *Every finite subgroup of $\pi_0 \text{Diff } M_g$ can be realized as a group of isometries of some hyperbolic structure on M_g .*

For $g \geq 2$ the map $\text{Diff } M_g \rightarrow \pi_0 \text{Diff } M_g$ is a homotopy equivalence, but it is unknown whether or not $\pi_0 \text{Diff } M_g$ can be lifted back into $\text{Diff } M_g$ as a subgroup. Theorem 1 solves the lifting problem for finite subgroups of $\pi_0 \text{Diff } M_g$.

We will call $\pi_0 \text{Diff } M_g$ the *modular group*² (Mod_g) since it is the natural generalization of the classical modular group for $g = 1$. Another common name is the *mapping class group*.¹ It is naturally isomorphic to the group of outer automorphisms of $\pi_1 M_g$. Mod_g acts on the Teichmüller space (T_g) of hyperbolic metrics on M_g (where two are considered equivalent if there is an isometry, isotopic to the identity, between them). T_g is homeomorphic to a $(6g - 6)$ -dimensional ball and the action of Mod_g on T_g is properly discontinuous and, except for an element of order two in genus two, faithful. Theorem 1 is equivalent to

THEOREM 2. *Every finite subgroup of Mod_g acting on T_g has a fixed point.*

For $g = 1$, T_g (for flat metrics of area 1) is the upper half plane, $\text{Mod}_g \approx GL(2, \mathbf{Z})$ acts by linear fractional transformations, and Theorem 2 is classical. In the higher genus case, Theorems 1 and 2 were known for G cyclic (Nielsen [4]), solvable (Fenchel [1]), and in many other special cases (see Zieschang [6]). Kravetz [2] gave a proof of Theorem 2, but it was based on the false belief that T_g , equipped with Teichmüller's metric, has negative curvature (see Masur [3]).

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² We make no restriction to the orientation-preserving case, however.

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Let S denote the set of isotopy classes of simple closed curves on M_g . If $\phi \in S$ then on any hyperbolic surface there is a unique geodesic isotopic to ϕ . Let $l_\phi: T_g \rightarrow \mathbf{R}_+$ denote the function which assigns to the point $x \in T_g$ the geodesic length of ϕ w.r.t. the metric x . When ϕ is a finite set of $\phi_i \in S$, let l_ϕ denote the sum $\sum_{\phi_i \in \phi} l_{\phi_i}$. Mod_g acts on S so let $G\phi$ denote the orbit of ϕ under the finite group $G \subset \text{Mod}_g$. Then the function $l_{G\phi}$ is G -invariant. The idea of the proof of Theorem 2 is to show that, if the set $G\phi$ is large enough, then $l_{G\phi}$ realizes a unique minimum in T_g . Since $l_{G\phi}$ is G -invariant, this minimum is a fixed point for the group G .

DEFINITION. A collection $\phi = \{\phi_i\} \in S$ is said to *fill up* M_g if, when the ϕ_i are isotoped to have a minimal number of pairwise intersections, then every component of $M_g - \bigcup \phi_i$ is topologically a disk. Equivalently, every $\tau \in S$ must intersect at least one of the ϕ_i when they are in this minimal position.

PROPOSITION 1. *If $\phi = \{\phi_i\} \in S$ fills up M_g , then $l_\phi: T_g \rightarrow \mathbf{R}_+$ has a unique minimum in T_g .*

In order to prove Proposition 1 we study the behavior of the function l_ϕ along certain horocycle-like paths in T_g . The prototype for such a path is the 1-parameter family of surfaces gotten by choosing a simple geodesic on a hyperbolic surface, cutting along it, and glueing it back together with a left-hand twist of distance t . For each $t \in \mathbf{R}_+$ the result is a new hyperbolic structure on M_g .

In general one can define a 1-parameter family of deformations by sheering along a *geodesic lamination with transverse measure* μ (g.l. μ . for short) (see Thurston [5]). These deformations were invented by Thurston who named them *left earthquakes*. A path in T_g determined by earthquaking along the g.l. μ . ν will be denoted by h_ν . The key step in proving Proposition 1 is

PROPOSITION 2. *The function l_ϕ is convex along h_ν . It is strictly convex iff the intersection number of ϕ with ν is nonzero.*

Finally we need to know that every two points in T_g are connected by some h_ν . This fact is due to Thurston (unpublished) and will appear in later chapters of [5].

THEOREM 3 (THURSTON). *For any $x, y \in T_g$ there is a unique left earthquake taking x to y .*

When ϕ fills up M_g , it is not hard to see that l_ϕ is a proper function. Using this fact, together with Theorem 3 and Proposition 2, it is easy to prove Proposition 1 and hence Theorem 2.

We remark that the methods used here work for a general Teichmüller space, when the hyperbolic manifold is allowed to have geodesic boundary or cone-like singularities and that the theorems were stated for closed surfaces only

to avoid the details involved in defining these structures and their modular groups. The nonorientable case can be handled by passing to the oriented double cover.

Finally, the Nielsen Problem is equivalent to describing when a finite extension of a surface group is again a surface group (in the generalized sense of a surface with cone-like singularities, etc.) and provides some new information about the homotopy type of 3-manifolds which have finite covering spaces which are Seifert manifolds. These subjects will be deferred to a format with more space.

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