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THE LANGLANDS CONJECTURE FOR Gl_2 OF A LOCAL FIELD

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Let F be a p-field and let $W(F)$ be the absolute Weil group of G. Let $\mathcal{A}_n(F)$ be the set of (equivalence classes of) continuous semisimple n-dimensional complex representations of $W(F)$ and let $\mathcal{A}(Gl_n(F))$ be the set of (equivalence classes of) irreducible admissible representations of $Gl_n(F)$. By local classfield theory there is a natural bijection between the sets $\mathcal{A}_1(F)$ and $\mathcal{A}(Gl_1(F))$, this latter set being just the set of quasi-characters of the multiplicative group $F^\times$ of F; we observe the convention of using this bijection to identify one-dimensional representations of $W(F)$ with quasi-characters of $F^\times$.

It is a conjecture of Langlands [JL] that there should exist a bijection $\sigma \longrightarrow \pi(\sigma)$ between $\mathcal{A}_2(F)$ and the subset of nonspecial representations in $\mathcal{A}(Gl_2(F))$, this bijection having the following properties.

1. For χ in $\mathcal{A}_1(F)$, $\pi(\sigma \otimes \chi) = \pi(\sigma) \otimes \chi \circ \det$.

2. The one-dimensional representation $\det \sigma$ should (under our convention) be the central character of $\pi(\sigma)$.

3. $L(\sigma) = L(\pi(\sigma))$; $\epsilon(\sigma) = \epsilon(\pi(\sigma))$ where L, ϵ are the *local factors* associated to σ and $\pi(\sigma)$ [JL] with respect to some fixed character of F^+.

In case the representation σ in $\mathcal{A}_2(F)$ is reducible or imprimitive (i.e., induced from a proper subgroup of $W(F)$) the existence of $\pi(\sigma)$ is demonstrated in [JL]; in particular, this verifies the conjecture in case $p \neq 2$.

In case $p = 2$, Yoshida [Y] and Ree [R] have shown the existence of $\pi(\sigma)$ for certain primitive representations σ and Tunnell [T] has shown that the map $\sigma \longrightarrow \pi(\sigma)$ is a bijection given that the existence of $\pi(\sigma)$ has already been established for all σ in $\mathcal{A}_2(F)$, thus establishing the validity of the conjecture for $F = \mathbf{Q}_2$ as well as for fields F of residual characteristic two which contain the cube roots of unity.

We have recently verified the existence of $\pi(\sigma)$ for any primitive representation σ of $\mathcal{A}_2(F)$ and we have shown that the map $\sigma \longrightarrow \pi(\sigma)$ is indeed a bijection with the properties described above. We give here a sketch of our methods; a more detailed description of our results will appear elsewhere.

1. As above, let F be a p-field, $p = 2$ and let σ be a primitive two-dimensional representation of $W(F)$. Then [W] there exists a unique extension $K = K(\sigma)$

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of F, galois with galois group $\Gamma_{K/F}$ either A_3 or S_3, such that the restriction, σ_K of σ to $\Gamma_{K/F}$ is imprimitive. We call $K(\sigma)$ the splitting field of σ. Given such a galois extension K/F we denote by $\mathbb{A}_2(K/F)$ the set of representations in $\mathbb{A}_2(F)$ with splitting field K.

Fix a character τ of F^+ and for any extension L/F, set $\tau_{L/F} = \tau \circ \mathrm{Tr}_{L/F}$.

LEMMA 1.1. *If σ is in $\mathbb{A}_2(K/F)$ then if $\Gamma_{K/F} \cong A_3$, $\epsilon(\sigma_k, \tau_{K/F}) = (\epsilon(\sigma, \tau))^3$ while if $\Gamma_{K/F} \cong S_3$ and H is any intermediate field to K/F with $[H: F] = 3$ then $\epsilon(\sigma_H, \tau_{H/F}) = \zeta(\epsilon(\sigma \otimes \omega, \tau))^3$ where ζ is a cube root of unity and ω is the nontrivial character of $F^\times$ corresponding to the quadratic unramified extension F_0/F.*

LEMMA 1.2. *Let $\mathfrak{A}$ be the field obtained by adjoining all nth roots of unity, $3 \nmid n$, to $\mathbf{Q}$. Let σ be a primitive two-dimensional representation of $W(F)$. Then*

1. *There exists a one-dimensional representation χ of $W(F)$ such that*
   (i) *$f(\sigma \otimes \eta) \geqslant f(\sigma \otimes \chi)$ for all η in $\mathbb{A}_1(F)$ where $f(\sigma)$ is the exponent of the Artin conductor of σ* **[S]**;
   (ii) $\det(\sigma \otimes \chi)$ *has values in $\mathfrak{A}$.*
2. *If χ is chosen with the above properties, then $\epsilon(\sigma \otimes \chi, \tau)$ lies in $\mathfrak{A}$.*

2. Let $G_F = Gl_2(F)$ and let π be an irreducible admissible supercuspidal **[JL]** representation of G_F. Then π will be called *unramified* if it may be induced from the subgroup $Z \cdot Gl_2(\mathcal{O}_F)$ where Z is the center of G_F and *ramified* otherwise. (It should be noted that this terminology is nonstandard; π is generally called unramified if its conductor is $\mathcal{O}_F$.)

LEMMA 2.1. *An irreducible supercuspidal representation π of G_F is unramified if and only if $\pi = \pi(\sigma)$ for some two-dimensional representation σ of $W(F)$ which is induced from $W(F_0)$.*

Now let π be a ramified irreducible supercuspidal representation of G_F. Then **[K]** π may be induced from a one-dimensional representation of a subgroup of G_F and as such is determined by a ramified quadratic extension E/F, a quasi-character ρ of $E^\times$, a quasi-character χ of $F^\times$ and a character η of E^+ **[GK]**. Let K/F be tamely ramified. Then by lifting ρ to $EK^\times$ and χ to $K^\times$ through the norm and lifting η to EK^+ through the trace, we obtain a representation π_K of G_K which we call a *tame lift* of π to G_K.

LEMMA 2.2. *If $[K: F] = 3$ then*

$$\epsilon(\pi_K, \tau_{K/F}) = [\epsilon(\pi, \tau)]^3.$$

LEMMA 2.3. *Let K/F be galois with prime cyclic galois group $\Gamma_{K/F}$. Then a representation π of G_K is a tame lift if and only if π is fixed under $\Gamma_{K/F}$.*

LEMMA 2.4. *Let π be an exceptional representation of G_F; i.e., a supercuspidal irreducible representation not of the form $\pi(\sigma)$ for an imprimitive representation σ of Γ_F. Then there exists a unique extension K/F, galois with $\Gamma_{K/F}$ either A_3 or S_3, such that π_K is not exceptional.*

With π, K as above we call K the *splitting field* for π and let $\mathcal{A}(G_K, G_F)$ be the subset of $\mathcal{A}(G_F)$ consisting of representations whose splitting field is K.

LEMMA 2.5. *Let π be an irreducible supercuspidal representation of G_F. Then there exists a quasi-character χ of $F^\times$ such that $\pi \otimes \chi \circ \det$ has minimal conductor and the central character of $\pi \otimes \chi \circ \det$ takes values in $\mathfrak{A}$. With χ as above, $\epsilon(\pi \otimes \chi \circ \det, \tau)$ lies in $\mathfrak{A}$.*

LEMMA 2.6. *Let σ be an imprimitive two-dimensional representation of $W(F)$, let F_0/F be quadratic unramified and suppose that σ_{F_0} is irreducible. Then $[\pi(\sigma)]_{F_0} = \pi(\sigma_{F_0}) \otimes \omega \circ \det$ where ω is an unramified character of $F_0^\times$ and $\omega^2 = 1$.*

3. THEOREM. *Let F be a 2-field and let K/F be galois with galois group either A_3 or S_3. Then the map $\sigma \longrightarrow \pi(\sigma)$ is defined on $\mathcal{A}_2(K/F)$ and puts $\mathcal{A}_2(K/F)$ into bijection with $\mathcal{A}(G_K, G_F)$.*

PROOF (SKETCH). First let $\Gamma_{K/F} \cong A_3$. Pick σ in $\mathcal{A}_2(K/F)$ such that $f(\sigma) \leqslant f(\sigma \otimes \chi)$ for quasi-characters χ of $F^\times$ and such that $\det \sigma$ takes values in $\mathfrak{A}$. Then $\pi(\sigma_K)$ exists and is fixed by $\Gamma_{K/F}$. By Lemma 2.3, $\pi(\sigma_K) = \pi_K$ for some representation π in $\mathcal{A}(G_K, G_F)$ and one may pick π such that its central character takes values in $\mathfrak{A}$. With this choice of π, $\pi = \pi(\sigma)$. In fact, it follows immediately that $\det \sigma$ is the central character of π. Also, one has $\epsilon(\pi \otimes \chi \circ \det, \tau) = \epsilon(\sigma \otimes \chi, \tau)$ for quasi-characters χ whose values lie in $\mathfrak{A}$ by Lemmas 1.1 and 2.2 and for tamely ramified χ by a direct computation. One then deduces that $\epsilon(\pi \otimes \chi \circ \det, \tau) = \epsilon(\sigma \otimes \chi, \tau)$ for arbitrary χ whence $\pi = \pi(\sigma)$. By Lemma 1.2 one may construct $\pi(\sigma)$ for any σ in $\mathcal{A}_2(K/F)$.

In the same manner, using Lemmas 2.3, 2.4, 2.5, one may construct an inverse map $\pi \longrightarrow \sigma(\pi)$ from $\mathcal{A}(G_K, G_F)$ to $\mathcal{A}_2(K/F)$ thus demonstrating the theorem if $\Gamma_{K/F} \cong A_3$.

Now suppose that $\Gamma_{K/F} \cong S_3$, let H and F_0 be as in Lemma 1.1 and pick σ in $\mathcal{A}_2(K/F)$ as above. Then σ_H is imprimitive and by Lemma 2.6, $[\pi(\sigma_H)]_K$ is fixed by $\Gamma_{K/F}$. It follows (with some work) that there exists a representation π in $\mathcal{A}(G_K, G_F)$ such that $\pi_H = \pi(\sigma_H)$. Since $\det \sigma$ is the unique extension of $\det \sigma_H$ to $W(F)$ and since the cube roots of unity do not lie in F, it follows from Lemmas 1.1 and 2.2 that $\pi = \pi(\sigma)$. Just as above, one may construct an inverse to the map $\sigma \longrightarrow \pi(\sigma)$ thus verifying the theorem in case $\Gamma_{K/F} \cong S_3$.

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