A CHARACTER FORMULA FOR THE DISCRETE SERIES OF A SEMISIMPLE LIE GROUP

BY JORGE VARGAS

ABSTRACT. For a semisimple Lie group $G$, we provide an explicit formula for the discrete series characters $\theta_\lambda$ restricted to the identity component of a split Cartan subgroup, whenever the parameter lies in a so-called Borel-de Siebenthal chamber and $G$ has both a compact Cartan subgroup and a split Cartan subgroup.

Let $G$ be a connected semisimple Lie group with finite center. The discrete series of $G$ is, by definition, the set of equivalence classes of irreducible unitary representations $\pi$, such that $\pi$ occurs discretely in the left (or right) regular representation of $G$. According to Harish-Chandra [3], $G$ has a nonempty discrete series if and only if $G$ contains a compact Cartan subgroup. Thus we fix a compact Cartan subgroup $B \subseteq G$, and a maximal compact subgroup $K \subseteq G$ which contains $B$. Let $\mathfrak{g}, \mathfrak{f}, \mathfrak{b}$ be the Lie algebras of $G, K, B$, and $\mathfrak{g}^C, \mathfrak{f}^C, \mathfrak{b}^C$ their complexifications. Let $\Phi = \Phi(\mathfrak{g}^C, \mathfrak{b}^C)$ be the root system of $(\mathfrak{g}^C, \mathfrak{b}^C)$. A root $\alpha \in \Phi$ is called compact (respectively noncompact) if its root space lies in $\mathfrak{f}^C$ (respectively the orthogonal complement of $\mathfrak{f}^C$). The differentials of the characters of $B$ form a lattice $\Lambda \subseteq i\mathfrak{b}^*$ ($\mathfrak{b}^*$ = dual space of $\mathfrak{b}$). The killing form induces a positive definite inner product $\langle \ , \ \rangle$ on $i\mathfrak{b}^*$. An element $\lambda \in \Lambda$ is called nonsingular if $\langle \lambda, \alpha \rangle \neq 0$ for every $\alpha \in \Phi$. We set $W = W(G, B) = Weyl$ group of $B$ in $G$. Equivalently, $W$ can be described as the group generated by the reflection about the compact roots in $i\mathfrak{b}^*$.

In order to state Harish-Chandra's enumeration of the discrete series [3], we assume, without loss of generality, that $G$ is acceptable in the sense of Harish-Chandra. Then, for each nonsingular $\lambda \in \Lambda$, there exists exactly one tempered1 invariant eigendistribution $\theta_\lambda$ on $G$, such that

$$\theta_\lambda|_{B \cap G'} = (-1)^q \frac{\Sigma_{w \in \mathcal{W}} \text{sgn } w e^{\mathcal{W} \lambda}}{\Pi_{\alpha \in \Phi, \langle \alpha, \lambda \rangle > 0} (e^{\alpha/2} - e^{-\alpha/2})}.$$ 

Here $q = \frac{1}{2} \dim G/K$, and $G'$ = set of regular semisimple points in $G$. Every $\theta_\lambda$ is the character of a discrete series representation, and conversely. Moreover, $\theta_\lambda = \theta_\mu$ if and only if $\lambda$ belongs to the $W$-orbit of $\mu$.

1 A distribution $\theta$ on $G$ is tempered if it extends to the Schwartz space of rapidly decreasing functions [3].

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When $G/K$ is a hermitian symmetric space, there exist positive root systems in $\Phi$ such that the sum of two noncompact positive roots is never a root. If $\lambda$ is dominant with respect to such a positive root system, $\theta_\lambda$ is the character of one of the so-called holomorphic discrete series representations. In this special situation, S. Martens [5] and H. Hecht [4] have given explicit global formulas for the characters $\theta_\lambda$. Whether or not $G/K$ is hermitian symmetric, there exist positive root systems which satisfy the following condition [1]:

for each noncompact simple factor $G_i$ of $G$, there exists exactly one noncompact simple root $\beta_i$, and this root $\beta_i$ occurs at most twice in the highest root of $G_i$.

(Borel-de Siebenthal property). The problem of computing the discrete series characters $\theta_\lambda$ globally can be reduced, at least in principle, to the following rather special situation:

(a) $G$ is simple and has both a compact Cartan subgroup $B$ and a split Cartan subgroup $A$.

(b) Compute $\theta_\lambda$ restricted to the identity component of a split Cartan subgroup $A$.

(c) The system of positive roots $\Psi = \{\alpha \in \Phi | (\alpha, \lambda) > 0\}$ has the Borel-de Siebenthal property. (See Schmid [6].)

From now on, let $G$, $\lambda$, $\Psi$, $A$ be as in (a)-(c), and $d$ an inner automorphism of $g^C$ such that $d : b^C \sim a^C$.

We denote the identity component of $A$ by $A^\circ$ and define

$$C = \{ \exp X | X \in a, (\alpha, d^{-1}X) < 0 \text{ for all } \alpha \in \Psi \}.$$  

The closure of $C$ and its conjugates cover $A^\circ$. Our main result provides an explicit formula for the restrictions of $\theta_\lambda$ to $A^\circ$. This formula involves a particular element $t$ of the Weyl group of $(g^C, b^C)$, whose description we defer until later.

**Theorem.** Let $W_U$ be the subgroup of $W$ generated by the compact simple roots for $\Psi$. Then

$$\theta_\lambda|_C = (-1)^q \frac{|W|}{|W_U|} \frac{\sum_{w \in W} \text{sgn}(tw) e^{t w \lambda}}{\prod_{\alpha \in \Psi} (e^{\alpha/2} - e^{-\alpha/2})} \circ d^{-1}$$

Let $\beta$ be the noncompact simple root for $\Psi$. Then $\beta$ is as long as or longer than any noncompact root, and the system $\Phi'$ of the roots in $\Phi$ orthogonal to $\beta$ has at most three irreducible components. Moreover if $\Phi'$ has more than one connected component, then all but perhaps one are of type $A_1$, and all the $A_1$-type components consist of noncompact roots. Two roots $\alpha_1, \alpha_2 \in \Phi$ are said to be strongly orthogonal if $\alpha_1 \pm \alpha_2 \notin \Phi$. For any strongly orthogonal subset $S \subset \Phi$ consisting of noncompact roots, we set $\Phi_S = \mathbb{Q}\text{-linear span of } S$ in $\Phi$. 

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Lemma 1. Each irreducible component of $\Psi \cap \Phi_S$ has the Borel-de Siebenthal property.

We now define a family of sub-root systems $\Phi = \Phi^0, \Phi^1, \ldots, \Phi^m$ inductively, as follows: $\Phi^{i+1}$ is the set of roots in $\Phi^i$ orthogonal to the noncompact simple roots for $\Psi \cap \Phi^i$, until the process stops. Let $S_0$ be the set consisting of all positive noncompact roots that are simple roots for some $\Psi \cap \Phi^i, 0 \leq i \leq m$.

Lemma 2. (a) $S_0$ is a strongly orthogonal set which spans $\Phi$ over $\mathbb{Q}$.
(b) $S_0$ contains at most one short root.

The next proposition describes the element $t$ of the Weyl group of $(\mathfrak{g}^C, \mathfrak{h}^C)$ which was used in the statement of the main result.

Proposition. There exists a unique $t$ in the Weyl group of $(\mathfrak{g}^C, \mathfrak{h}^C)$ such that
(1) $t$ is a product of reflections about roots in $S_0$.
(2) If $\beta$ occurs twice in the highest root, then $t \neq 1$.
(3) $t$ takes any long simple root into a noncompact root.
(4) $(tw\lambda, \mu) \geq 0$ for every $w \in W_U$ and $\lambda, \mu$ dominant integral with respect to $\Psi$.

From the Proposition, one can deduce the following properties of $t$:
(a) If $\alpha_1, \alpha_2$ are two adjacent long simple roots, then $\text{sgn } t\alpha_1 \neq \text{sgn } t\alpha_2$.
(b) Assume $\beta$ occurs twice in the highest root. Then $t$ fixes any short root in $S_0$.
(c) If $S_0$ does not contain short roots, $t\alpha = \alpha$ for any short simple root $\alpha$.
(d) Again under the assumption that $\beta$ occurs twice in the highest root, if $S_0$ does contain short roots, $t\alpha$ is noncompact for any short simple root $\alpha$.

The proof of the theorem proceeds by induction on the dimension of $G$. The crux of the matter is to verify the consistency of our formula with Harish-Chandra's matching conditions [2]. Details will appear elsewhere.

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References