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A CHARACTER FORMULA FOR THE DISCRETE SERIES OF A SEMISIMPLE LIE GROUP

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ABSTRACT. For a semisimple Lie group G, we provide an explicit formula for the discrete series characters θ_λ restricted to the identity component of a split Cartan subgroup, whenever the parameter lies in a so-called Borel-de Siebenthal chamber and G has both a compact Cartan subgroup and a split Cartan subgroup.

Let G be a connected semisimple Lie group with finite center. The discrete series of G is, by definition, the set of equivalence classes of irreducible unitary representations π, such that π occurs discretely in the left (or right) regular representation of G. According to Harish-Chandra [3], G has a nonempty discrete series if and only if G contains a compact Cartan subgroup. Thus we fix a compact Cartan subgroup $B \subset G$, and a maximal compact subgroup $K \subset G$ which contains B. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{b}$ be the Lie algebras of G, K, B, and $\mathfrak{g}^{\mathbf{C}}$, $\mathfrak{k}^{\mathbf{C}}$, $\mathfrak{b}^{\mathbf{C}}$ their complexifications. Let $\Phi = \Phi(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ be the root system of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$. A root $\alpha \in \Phi$ is called compact (respectively noncompact) if its root space lies in $\mathfrak{k}^{\mathbf{C}}$ (respectively the orthogonal complement of $\mathfrak{k}^{\mathbf{C}}$). The differentials of the characters of B form a lattice $\Lambda \subset i\mathfrak{b}^*$ ($\mathfrak{b}^*$ = dual space of $\mathfrak{b}$). The killing form induces a positive definite inner product $(\ ,\ )$ on $i\mathfrak{b}^*$. An element $\lambda \in \Lambda$ is called nonsingular if $(\lambda, \alpha) \neq 0$ for every $\alpha \in \Phi$. We set $W = W(G, B)$ = Weyl group of B in G. Equivalently, W can be described as the group generated by the reflection about the compact roots in $i\mathfrak{b}^*$.

In order to state Harish-Chandra's enumeration of the discrete series [3], we assume, without loss of generality, that G is acceptable in the sense of Harish-Chandra. Then, for each nonsingular $\lambda \in \Lambda$, there exists exactly one tempered[1] invariant eigendistribution θ_λ on G, such that

$$\theta_\lambda|_{B\cap G'} = (-1)^q \frac{\Sigma_{w\in W} \operatorname{sgn} w e^{w\lambda}}{\Pi_{\alpha\in\Phi,(\alpha,\lambda)>0} (e^{\alpha/2} - e^{-\alpha/2})}.$$

Here $q = \frac{1}{2} \dim G/K$, and G' = set of regular semisimple points in G. Every θ_λ is the character of a discrete series representation, and conversely. Moreover, $\theta_\lambda = \theta_\mu$ if and only if λ belongs to the W-orbit of μ.

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[1] A distribution θ on G is tempered if it extends to the Schwartz space of rapidly decreasing functions [3].

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When G/K is a hermitian symmetric space, there exist positive root systems in Φ such that the sum of two noncompact positive roots is never a root. If λ is dominant with respect to such a positive root system, θ_λ is the character of one of the so-called holomorphic discrete series representations. In this special situation, S. Martens [5] and H. Hecht [4] have given explicit global formulas for the characters θ_λ. Whether or not G/K is hermitian symmetric, there exist positive root systems which satisfy the following condition [1]:

> for each noncompact simple factor G_i of G, there exists exactly one noncompact simple root β_i, and this root β_i occurs at most twice in the highest root of G_i.

(Borel-de Siebenthal property). The problem of computing the discrete series characters θ_λ globally can be reduced, at least in principle, to the following rather special situation:

(a) G is simple and has both a compact Cartan subgroup B and a split Cartan subgroup A.

(b) Compute θ_λ restricted to the identity component of a split Cartan subgroup A.

(c) The system of positive roots $\Psi = \{\alpha \in \Phi \mid (\alpha, \lambda) > 0\}$ has the Borel-de Siebenthal property. (See Schmid [6].)

From now on, let G, λ, Ψ, A be as in (a)–(c), and d an inner automorphism of $\mathfrak{g}^{\mathbf{C}}$ such that $d: \mathfrak{b}^{\mathbf{C}} \xrightarrow{\sim} \mathfrak{a}^{\mathbf{C}}$.

We denote the identity component of A by A° and define

$$C = \{\exp X \mid X \in \mathfrak{a}, \langle \alpha, d^{-1}X \rangle < 0 \text{ for all } \alpha \in \Psi\}.$$

The closure of C and its conjugates cover A°. Our main result provides an explicit formula for the restrictions of θ_λ to A°. This formula involves a particular element t of the Weyl group of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$, whose description we defer until later.

THEOREM. *Let W_U be the subgroup of W generated by the compact simple roots for Ψ. Then*

$$\theta_\lambda|_C = (-1)^q \frac{|W|}{|W_U|} \frac{\Sigma_{w \in W} \operatorname{sgn}(tw) e^{tw\lambda}}{\Pi_{\alpha \in \Psi} (e^{\alpha/2} - e^{-\alpha/2})} \circ d^{-1}$$

Let β be the noncompact simple root for Ψ. Then β is as long as or longer than any noncompact root, and the system Φ' of the roots in Φ orthogonal to β has at most three irreducible components. Moreover if Φ' has more than one connected component, then all but perhaps one are of type A_1, and all the A_1-type components consist of noncompact roots. Two roots $\alpha_1, \alpha_2 \in \Phi$ are said to be strongly orthogonal if $\alpha_1 \pm \alpha_2 \notin \Phi$. For any strongly orthogonal subset $S \subset \Phi$ consisting of noncompact roots, we set $\Phi_S = \mathbf{Q}$-linear span of S in Φ.

LEMMA 1. *Each irreducible component of* $\Psi \cap \Phi_S$ *has the Borel-de Siebenthal property.*

We now define a family of sub-root systems $\Phi = \Phi^0, \Phi^1, \ldots, \Phi^m$ inductively, as follows: Φ^{i+1} is the set of roots in Φ^i orthogonal to the noncompact simple roots for $\Psi \cap \Phi^i$, until the process stops. Let S_0 be the set consisting of all positive noncompact roots that are simple roots for some $\Psi \cap \Phi^i$, $0 \leqslant i \leqslant m$.

LEMMA 2. (a) S_0 *is a strongly orthogonal set which spans* Φ *over* $\mathbf{Q}$.
(b) S_0 *contains at most one short root.*

The next proposition describes the element t of the Weyl group of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ which was used in the statement of the main result.

PROPOSITION. *There exists a unique t in the Weyl group of* $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{b}^{\mathbf{C}})$ *such that*
(1) *t is a product of reflections about roots in* S_0.
(2) *If* β *occurs twice in the highest root, then* $t \neq 1$.
(3) *t takes any long simple root into a noncompact root.*
(4) $(tw\lambda, \mu) \geqslant 0$ *for every* $w \in W_U$ *and* λ, μ *dominant integral with respect to* Ψ.

From the Proposition, one can deduce the following properties of t:
(a) If α_1, α_2 are two adjacent long simple roots, then $\operatorname{sgn} t\alpha_1 \neq \operatorname{sgn} t\alpha_2$.
(b) Assume β occurs twice in the highest root. Then t fixes any short root in S_0.
(c) If S_0 does not contain short roots, $t\alpha = \alpha$ for any short simple root α.
(d) Again under the assumption that β occurs twice in the highest root, if S_0 does contain short roots, $t\alpha$ is noncompact for any short simple root α.
The proof of the theorem procedes by induction on the dimension of G. The crux of the matter is to verify the consistency of our formula with Harish-Chandra's matching conditions [2]. Details will appear elsewhere.

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