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Brownian motion and classical potential theory, by Sidney C. Port and Charles J. Stone, Academic Press, New York, 1978, xii + 236 pp., $22.50.

This book does an excellent job of developing classical potential theory based on Brownian motion.

"Classical potential theory" means the theory of the Laplacian on a domain in E^n. For the purposes of this review one should think of "Brownian motion" as consisting of the entire following collection of objects: the Gauss kernel $p(t, x, y) = (2\pi t)^{-n/2} \exp(-|x - y|^2/2t)$, $t > 0$, x and y in E^n, with the corresponding semigroup $P_t f(x) = \int_{E^n} p(t, x, y) f(y)\, dy$ of operators on functions; a family $\{P^x; x \text{ in } E^n\}$ of probability measures on the space Ω of continuous paths $t \to X(t)$ from $[0, \infty)$ to E^n. The measures P^x have special properties such as (1) each P^x attributes mass 1 to the set of paths that start at the point x, (2) integrals over E^n are related to integrals over function space by formulas such as: $P_t f(x) = \int_\Omega f(X(t))\, dP^x$ and more complicated iterates of these. A probabilist would say simply that relative to each P^x the coordinate functions $X(t)$ on the sample space Ω form a time homogeneous Markov process with transition density $p(t, x, y)$ and initial position x.

Henceforth we will use the abbreviation $E^x Y$ for the integral $\int_\Omega Y\, dP^x$. Many useful aspects of the theory involve appropriately selecting Y and then developing properties of the function $x \to E^x Y$. For definiteness take $n = 3$ and define the potential of a function f or a measure μ by $Uf(x) = \int_0^\infty P_t f(x)\, dt$ and $\mu U(y) = \int_{E^3} \mu\,(dx)|x - y|^{-1}$, so that these are simply the familiar Newtonian potentials. On the other hand an extension of (2) above shows that $Uf(x) = E^x Y$ with $Y = \int_0^\infty f(X(t))\, dt$ and, more informally, that $\mu U(y)\, dy$ represents the expected length of time that the wandering path $t \to X(t)$ spends in the volume element dy about y if the initial position $X(0)$ is random with distribution μ. In either case the analytic object is expressed as an integral over function space.

We will give three examples to show how such representations illuminate aspects of classical potential theory. Of course all three are discussed thoroughly in the book under review. They all involve the following objects: a subset A of E^3, the (random) time T at which the path $t \to X(t)$ first meets the set A, and the place $X(T)$ at which the first meeting occurs.

(1) Dirichlet problem. If D is a region and f is a function on its boundary, then take A to be the boundary of D and set $h(x) = E^x f(X(T))$. It is fairly obvious that on D the function h has the mean value property (any small sphere about x in D must be hit by the path before it proceeds to the boundary A, while symmetry considerations dictate that the hitting distribution on the sphere will be uniform). And if x is near the boundary and the boundary is smooth then the hitting place $X(T)$ should, with high probability, be close to x. Thus at least if f is continuous, the right boundary behavior should hold and so h solves the Dirichlet problem with boundary data f. This seems like an appealing way to start a rigorous discussion of the complete problem.

(2) Maximum principle. The definition $Uf = \int_0^\infty P_t f\,dt$ and the semigroup properties of the operators P_t show that $P_t Uf = \int_t^\infty P_s f\,ds$ and this as a function space integral is $E^{\cdot}\int_t^\infty f(X(s))\,ds$. Some attention to measure theory shows that this formula is still valid if the fixed time t is replaced by the random time T already defined. The resulting formula is $E^x\int_T^\infty f(X(s))\,ds = \int \mu(dy)Uf(y)$ where μ is the distribution of the hitting place $X(T)$. In particular if f vanishes off the set A then the probability integral is equal to $E^x\int_0^\infty f(X(s))\,ds$ or Uf so in this case we have $Uf = \mu Uf$ with μ a measure carried by A. It is an immediate consequence that if f and g are positive functions with the property that Ug exceeds Uf on the support of f then Ug exceeds Uf everywhere. This is the maximum principle of classical potential theory. The probability argument is very direct.

(3) Balayage. If μ is a measure and A is a set, one looks for a measure ν which is concentrated on A and has the property that νU agrees with μU on A. (Once ν is shown to exist, the operation of passing from μ to ν is called balayage.) A discussion exactly like that of the maximum principle shows that the desired ν is simply the distribution of $X(T)$ if the initial position, $X(0)$, of the path is given the distribution μ. So again a "solution" to the problem is obtained quickly. Of course, much of this apparent brevity is a little misleading as we will indicate later.

Leaving the illustrations and returning to a more general description, the meaning of basing potential theory on probability is this: one uses the stochastic process (or the measures on function space) to define and analyze potential theoretic objects like superharmonic functions, balayage, regular points for the Dirichlet problem. Also one uses probability techniques like time reversal of the stochastic process to prove symmetry of the Green function or to establish properties of capacity. The book under review treats the most important case of the general theory. The general theory was given in 1957 by G. A. Hunt. He defined a potential operator as a positive linear transformation, U, of continuous functions on some space, satisfying the maximum principle of our second example. He showed that associated with each such operator is a Markov semigroup $\{P_t\}$ so that $U = \int_0^\infty P_t\,dt$ as in the case at hand. He then used the accompanying Markov process to derive and study a theory of potentials which is related to U in exactly the same way as ordinary potential theory in E^3 is related to the Newtonian potential operator. The "Brownian motion" case is what one gets when the Markov semigroup arises from the Gauss kernel or from some rather specific alteration of it necessary to handle potential theory on a proper subdomain of the space. In this case the probabilistic approach had been used with profit by Doob, Kac, Kakutani and others prior to the appearance of Hunt's general theory.

To appreciate the probability approach it is not necessary to learn general Markov process theory. Practically everything of importance already is significant when one derives classical potential theory from Brownian motion. Here the potential theory is familiar (harmonic functions, Newtonian capacity, etc.) while the probability is known well enough that extensive measure theoretic preliminaries are not required. The authors have taken full advantage of this somewhat limited context to make a readable and enjoyable presentation.

The book covers most of what is important in classical potential theory. For a definition of "importance" the reviewer used the book *Introduction to potential theory* by L. L. Helms. Almost every fact in Helms' book can be found in Port and Stone, and a comparison of the two books is interesting. For example, in the probability approach one writes down more or less immediately a solution of the Dirichlet problem in an arbitrary domain, while the classical approach starts more slowly with a ball and Poisson's integral formula. However by the time the treatment via probability is complete, including irregular domains, discontinuous boundary functions and exceptional boundary sets, it is just as long as the complete classical treatment. In fact the two books have the same length, so that some "conservation" principle is at work.

The listed prerequisites–knowledge of real variable theory plus a graduate level probability course–are more than adequate. It would have been feasible and appropriate to include exercises, especially since many people come to this subject considerably more familiar with one side than the other.

A moderate-sized book cannot contain everything. Some omissions on the potential theory side are the fine topology and the Martin boundary. Both topics would have fit in nicely: the probability approach leads to a very nice definition of "finely open" set, while the symmetry of the Green function eases the difficulty in proving the fundamental fact that the process converges to the Martin boundary. On the probability side the most important omissions are multiplicative and additive functionals, the latter being the sample function analogue of measures with finite potential.

In summary, this book is an attractive introduction to probabilistic potential theory. Readers who wish to learn important probablistic applications of the theory should next tackle the challenging *Diffusion processes and their sample paths* by Itô and McKean.

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Convexity and optimization in Banach spaces, by V. Barbu and Th. Precupanu, Sÿthoff & Noordhoff International Publishers, Alphen aan den Rijn, The Netherlands, 1978, xi + 316 pp.

The objective in studying abstract optimization problems is the development of a comprehensive theory that will contain specific optimization problems as special cases. Today, the mainstream of research in this area results from the confluence of developments in optimal control theory and in mathematical programming. Optimal control theory, in turn, encompasses much of the classical calculus of variations.

The calculus of variations originated in 1697 with the solutions of the brachistochrone problem by John and James Bernoulli. From then onward, the calculus of variations was a central and vital subject in mathematics. By