tage of this somewhat limited context to make a readable and enjoyable presentation.

The book covers most of what is important in classical potential theory. For a definition of "importance" the reviewer used the book *Introduction to potential theory* by L. L. Helms. Almost every fact in Helms' book can be found in Port and Stone, and a comparison of the two books is interesting. For example, in the probability approach one writes down more or less immediately a solution of the Dirichlet problem in an arbitrary domain, while the classical approach starts more slowly with a ball and Poisson's integral formula. However by the time the treatment via probability is complete, including irregular domains, discontinuous boundary functions and exceptional boundary sets, it is just as long as the complete classical treatment. In fact the two books have the same length, so that some "conservation" principle is at work.

The listed prerequisites–knowledge of real variable theory plus a graduate level probability course–are more than adequate. It would have been feasible and appropriate to include exercises, especially since many people come to this subject considerably more familiar with one side than the other.

A moderate-sized book cannot contain everything. Some omissions on the potential theory side are the fine topology and the Martin boundary. Both topics would have fit in nicely: the probability approach leads to a very nice definition of "finely open" set, while the symmetry of the Green function eases the difficulty in proving the fundamental fact that the process converges to the Martin boundary. On the probability side the most important omissions are multiplicative and additive functionals, the latter being the sample function analogue of measures with finite potential.

In summary, this book is an attractive introduction to probabilistic potential theory. Readers who wish to learn important probablistic applications of the theory should next tackle the challenging *Diffusion processes and their sample paths* by Itô and McKean.

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Convexity and optimization in Banach spaces, by V. Barbu and Th. Precupanu, Sÿthoff & Noordhoff International Publishers, Alphen aan den Rijn, The Netherlands, 1978, xi + 316 pp.

The objective in studying abstract optimization problems is the development of a comprehensive theory that will contain specific optimization problems as special cases. Today, the mainstream of research in this area results from the confluence of developments in optimal control theory and in mathematical programming. Optimal control theory, in turn, encompasses much of the classical calculus of variations.

The calculus of variations originated in 1697 with the solutions of the brachistochrone problem by John and James Bernoulli. From then onward, the calculus of variations was a central and vital subject in mathematics. By the mid 1940s, however, the study of single integral problems in the calculus of variations appears to have reached a dead end. In the late 1950s, optimal control theory appeared on the scene. Its birth can be attributed to the work of Pontryagin, Boltyanskii, Gamkrelidze, and Mishchencko, which was collected in **[6]**. Its popularity and success can be attributed to its inclusion of problems in diverse areas of application that are variational in nature, but which at first glance do not appear to fall under classical variational theory.

In essence, optimal control problems have the following form. Let t denote time. Let u be a "control" function defined on a time interval $[t_0, t_1]$ with values $u(t)$ in E^m and satisfying constraints of the form $u(t) \in \Omega(t)$ in $[t_0, t_1]$. Here $t \to \Omega(t)$ is a mapping from $[t_0, t_1]$ to subsets of E^m. Let x be a vector in E^n. Let φ be a "trajectory" satisfying the "state" differential equation $dx/dt = f(t, x, u(t))$ and "end conditions" $(t_0, \varphi(t_0), t_1, \varphi(t_1)) \in \mathfrak{B}$, where $\mathfrak{B}$ is some preassigned set. It is required to choose u and φ so that

$$J(\varphi, u) = g(t_0, \varphi(t_0), t_1, \varphi(t_1)) + \int_{t_0}^{t_1} f^0(t, \varphi(t), u(t))\, dt$$

is minimized, where g and f^0 are real valued functions. The classical problem of Bolza in the calculus of variations can be written as a control problem and the control problem can be written as a Bolza problem which differs from the classical one only in that there are constraints on the derivatives.

The principal contribution of Pontryagin et al. was the "Pontryagin maximum principle", which is a set of necessary conditions satisfied by a solution (φ^*, u^*) of the optimal control problem. When specialized to the Bolza problem, the Pontryagin maximum principle is a combined statement of the Euler-Lagrange equations (multiplier rule), the Weierstrass condition, and the transversality conditions. In their proof of the maximum principle, Pontryagin and his collaborators showed that optimality implies the separation of certain convex sets and that the maximum principle follows from this separation. Approximately twenty years earlier, McShane **[4]** used convexity in a similar way in his proof of the Weierstrass condition for the Lagrange problem; the Pontryagin et al. constructions were modifications of McShane's. Their proof of the maximum principle, however, popularized the use of the separation of appropriate convex sets to characterize optimality.

Mathematical programming began to develop in the early 1950s as an extension of linear programming, a subject whose own development began in the late 1940s. As with control theory, interest in mathematical programming was generated by problems in areas of application. The mathematical programming problem in R^n has the following form. Let x be a vector in R^n, let f be a real valued mapping defined on a subset X_0 of R^n, let g be a mapping from X_0 to R^k, and let h be a mapping from X_0 to R^m. Find an x^* in X_0 that minimizes f subject to the constraints $g(x) = 0$, $h(x) \leqslant 0$. If f, g, and h are affine, then we have a linear programming problem.

In 1951 Kuhn and Tucker **[3]** presented necessary conditions that must be satisfied by a solution of the programming problem in which $g \equiv 0$ and showed that if f and h are convex and $g \equiv 0$ then the programming problem is equivalent to a certain saddle value problem. One necessary condition was an extension of the Lagrange multiplier rule for the problem of minimizing f

subject to equality constraints $g = 0$. Although similar results had been obtained earlier by Karush [2] and John [1], the Kuhn-Tucker paper appears to have been the catalyst for further research. The Kuhn-Tucker proof made essential use of a theorem of Farkas concerning systems of linear equalities. The Farkas theorem was soon recognized to be a corollary of a theorem concerning polars of closed convex cones in euclidean space. Convexity and separation of appropriate convex cones were thus seen to underlie the necessary conditions for programming problems, as well as for control problems.

By the early to mid 1960s, the similarities in the approaches to necessary conditions for control problems and programming problems led to the study of abstract, or general, programming problems. These problems have the following structure. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let a positive cone $\mathcal{P}$ be defined in $\mathcal{Y}$. Let f be a real valued function defined on $\mathcal{X}$, let g and h be mappings from $\mathcal{X}$ to $\mathcal{Y}$ and let θ denote the zero vector in $\mathcal{Y}$. Let $\mathcal{X}_0$ be a subset of $\mathcal{X}$. Find an element x^* in $\mathcal{X}_0$ that minimizes f subject to the constraints $g(x) = \theta$, $h(x) \in \mathcal{P}$. The objective of the abstract formulation is to obtain a problem of sufficient generality to encompass a wide variety of specific control problems and programming problems, yet have sufficient structure to obtain necessary conditions that will translate into meaningful necessary conditions for the specific problems. This effort was originated independently by Dubovitskii and Milyutin, Halkin, Hestenes, and Neustadt. This work was anticipated to varying degrees by Lyusternik and by Goldstine in the 1930s and by Hurwicz, in 1958. A critique, a detailed historical account and complete references can be found in the "Notes and Historical Comments" of [5].

In the latter half of the 1960s a sightly different direction of research was initiated by R. T. Rockafellar. Almost all of the work on necessary and sufficient conditions had assumed that the functions involved possessed derivatives, however general or weak. In [7] and [8] Rockafellar studied programming problems over infinite dimensional spaces in which he assumed that the functions occurring were convex, but not necessarily differentiable at all points. He characterized the solutions of such problems in terms of subgradients and subdifferentials and extended the Fenchel duality theory for convex functions and programming problems in R^n to such problems. His work also extended Fenchel's work in R^n.

The importance of duality lies in the following considerations. If the original problem is a minimization problem, the dual problem is a maximization problem whose supremum is less than or equal to the infimum of the original problem. In some situations equality holds. Thus the dual problem can be used to obtain lower bounds for the minimum value of the original problem, and sometimes it can be used to solve the original problem. The two types of duality in current use are the minimax duality and the conjugate function duality.

In the early 1970s, Rockafellar [8], [9], obtained necessary and sufficient conditions and duality theorems for optimal control problems in which the functions involved were assumed to be convex but not differentiable. Rockafellar called these problems "convex problems of Bolza". The maximum principle is now stated in terms of subgradients and subdifferentials. The duality involved is again an extension of the Fenchel duality, rather than the minimax duality.

The book under review is concerned to a large extent with an exposition and elaboration of Rockafellar's work on abstract programming problems and with an extension of Rockafellar's results on the convex optimal control problem in R^n to convex optimal control problems in Hilbert space. In their preface, the authors state that it is their intention to make the book self contained. To achieve this end, their first chapter is devoted to a survey of results in functional analysis that are useful in abstract optimization theory. The second chapter is a survey of convexity and of convex functions defined on real Banach spaces. The subdifferential of a convex function is introduced and its properties are developed. The third chapter is devoted to optimality conditions and to a discussion of duality for convex abstract programming problems. As already noted, much of the material presented here originated with Rockafellar. The final chapter is concerned chiefly with the generalization of Rockafellar's work on the convex problem of Bolza to convex optimal control problems in Hilbert space. All chapters, except the first, have informative bibliographical notes, which relate the material in the chapter to other work.

As advertised, the first three chapters do constitute a self contained, fairly complete exposition of an established research area in optimization theory. The contribution of the last chapter will depend on the degree of future application of the material.

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