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Semigroups and combinatorial applications, by Gerard Lallement, Pure and Applied Mathematics Series, Wiley, New York, 1979, xii + 376 pp., $27.50.

One way in which algebra differs from analysis is its closeness to the axioms. Whereas real numbers require a short lecture course to be properly defined, groups need only three axioms, and a theory of remarkable depth and richness springs from this modest beginning. Semigroups need even fewer axioms, and although their theory does not, and probably never will, rival group theory in importance, it has certainly seen a considerable development in recent years. If in some circles the subject has gained a reputation for dullness, this may be explained (though not justified) by the fact that it is possible to write on semigroups with little previous knowledge and as a consequence a large mass of only very loosely related facts had been accumulated. This impression was reinforced by the appearance of Clifford-Preston [**1**] which faithfully recorded these facts and, though excellent as a reference, was much less well suited for continuous reading. The book by Howie [**3**] gives a readable introduction to the subject but confines itself to what might be called 'pure' semigroup theory, leaving such important topics as codes and languages untouched. In Howie's defence it might be said that there are now excellent books on both these topics, but it would surely be desirable to have a connected treatment of all these matters between one pair of covers. Eilenberg's volumes [**2**] deal with both semigroups and languages in a profound and original way which will clearly have a decisive influence on the subject, but it is not (and was not intended to be) a survey of what was known. Such a survey, if really exhaustive, would probably be unreadable, but one might hope for an account stressing the interaction between semigroups and language theory. This is provided by Lallement's book which aims "to present those parts of the theory of semigroups that are directly related to automata theory, algebraic linguistics and combinatorics". Thus it fills a real gap in the current literature, and fills it rather well. The first 100 pages are pure semigroup theory, the next 200 or so deal with free semigroups, codes and languages and the rest is devoted to combinatorial questions related to (a) Burnside's problem and (b) MacMahon's Master Theorem.

The history of semigroups should be easy to write, since most of it has taken place in the last 50 years and of the references in this book well over two-thirds date back less than twenty years. Almost the first step was the theorem of Suschkewitsch (1928) describing the structure of the minimal ideal in a finite semigroup, later generalized by D. Rees (1940) to completely simple semigroups (i.e. simple semigroups with minimal left and right ideals). Let A, B be any sets, G a group and $P = (p_{\beta\alpha})$ a $B \times A$ matrix with entries in G, then the *Rees matrix semigroup* over G with *sandwich matrix P* consists of all $A \times B$ matrices with a single entry in G and the rest 0 or undefined, say $[\alpha, g, \beta]$, with multiplication $[\alpha, g, \beta][\alpha', g', \beta'] = [\alpha, gp_{\beta\alpha'}g', \beta']$. Now the Rees-Suschkewitsch theorem states that any such matrix semigroup is completely simple and conversely every completely simple semigroup is of this form. All this is carefully presented, with the modification needed to include semigroups with zero, and numerous examples and applications, e.g. to block designs.

For more general semigroups less is known. Two elements a, b of a semigroup S are said to be R-, L- or J- equivalent if the right, left or two-sided ideals generated by a and b coincide. The equivalences, R, L, J and $D = R \circ L = L \circ R$, $H = R \cap L$ are Green's relations (1951). If one represents R-classes by horizontal, L-classes by vertical strips, the intersections are just the H-classes. This is the egg-box picture, which throws some light on the structure of S, though its main use is for finite semigroups. One of the basic structure theorems for groups is the Jordan-Hölder theorem; on the first meeting this seems to give a very loose classification–many different groups can have the same composition series–but it is quite precise compared with the corresponding result in semigroups. This is the prime decomposition theorem due to Krohn and Rhodes (1965), which associates with every finite monoid M (= semigroup with 1) a set of 'prime divisors' (certain simple groups and 1- or 2-element semigroups) from which M can be built up by taking wreath products and repeated extensions. The proof given is the author's own, based on one by Krohn, Rhodes and Tilson.

To describe languages, free monoids are needed. A free monoid on a set X, the 'alphabet', is defined by the universal mapping property, but it is much more easily described as the set X^* of all strings (or words) on X with juxtaposition as multiplication. To describe free groups correspondingly is not quite so trivial, because there one has to cope with cancellation of xx^{-1} and $x^{-1}x$. Any free generating set of a free submonoid is called a *code*, but the most useful codes are *prefix* codes for which no word is prefix of another word, e.g. y, xy is a prefix code, but x, xy is not (both sets are free). Whereas checking for prefix codes is a simple matter of inspection, general codes may be harder to recognize, though an algorithm exists (Sardinas-Patterson) which the author derives and illustrates. This leads rather naturally to algorithmic questions such as the word problem for monoids and groups, and a sketch proof (taking for granted some fairly plausible properties of recursive functions) of the existence of a group with unsolvable word problem.

We now come to languages. Formally, a language is a subset of a free monoid, but one normally imposes conditions on the subset to obtain interesting classes of languages. This can be done in three essentially different ways, using grammars, machines or formal power series, and important classes can be described in each of these three ways; thus regular languages correspond to rational power series (Kleene-Schützenberger) and context-free languages correspond to algebraic power series (Schützenberger). The author tends to concentrate here on the description by machines (automata); a regular language L on a finite alphabet X is completely described by a simple automaton (= finite-state acceptor), i.e. a finite set A with an action of the free monoid X^*, and this action defines a finite monoid, the *syntactic* monoid of L. The author examines in detail the languages whose syntactic monoids have the property that all their subgroups are abelian, and also briefly considers the case of soluble subgroups. A separate chapter is devoted to the regular languages generated by prefix codes.

The Burnside problem for matrix groups was solved by Schur (1911) who proved that a finitely generated periodic matrix group over C is finite. This was generalized to arbitrary fields by Kaplansky (1965) and to semigroups by McNaughton and Zalcstein (1975). The author presents the latter result using methods of G. Jacob, and in the opposite direction gives examples of infinite sequences without repeats (Morse-Hedlund sequence, though his actual examples follow J. Leech and F. Dejean).

The final chapter is devoted to Foata's theory of "rearrangement monoids", which leads to a quick and transparent proof of MacMahon's Master Theorem, and there are some illustrations from matching problems. The topic is closely related to factorizations in free monoids and bases in free Lie algebras, and these connexions are briefly sketched.

The writing is terse but clear; the many worked illustrations and exercises are particularly useful. While the first four chapters form an excellent introduction to semigroup theory, the account of language theory that follows it is inevitably somewhat biased, being seen from the viewpoint of semigroups. This would make it rather hard as a first introduction, but for anyone with even a nodding acquaintance of language theory it provides an interesting attempt to present an integrated account of these topics. It is only partly successful because the author has perhaps spent a disproportionate amount of space on rather recondite properties of languages because they happen to involve semigroups. However, this is more than made good by the many simplifications, and new applications of semigroup theory. There are one or two slips, of logic (isomorphism of composition series is so defined as to be nonreflexive), spelling (p. 91 preversed, p. 333 envelopping), a curious confusion of left and right on p. 4 and a definition of generating set which quite outdoes Bourbaki (see also the rather bizarre definition of mathematical problem, p. 127). But these are trifles which do not in any way detract from a highly readable book, which will surely help to kill the myth that there are no exciting results in semigroup theory.

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The representation theory of the symmetric groups, by G. D. James, Lecture Notes in Mathematics, Volume 682, Springer-Verlag, Berlin and New York, 1978, vi + 156 pp.

Since every finite group G is a subgroup of some symmetric group $\mathfrak{S}_n$, consisting of all the $n!$ permutations on n objects, the representations of $\mathfrak{S}_n$