reader is already familiar with the additional phenomena and intricacies which arise when general boundary conditions and functionals are considered.

A sequel to the present book would be welcome. One which develops analogous results for general functionals and general boundary conditions including the bounded state variable case. It would be instructive also to develop a similar theory using generalized controls for problems whose solutions are not minimizing but become minimizing when suitable isoperimetric conditions are adjoined. Many problems in mechanics and in the theory of geodesics are of this nature. It should be noted that most examples appearing in the literature deal only with ordinary controls. Perhaps this is because their solutions involve only ordinary controls. There are, of course, examples which require generalized controls. As has been pointed out by E. J. McShane there is a need for developing techniques for solving typical examples from the point of view of generalized controls even when the solutions are given by ordinary controls. The remarks of McShane are given in *The calculus of variations from the beginning through optimal control theory* appearing in Optimal Control and Differential Equations, Academic Press, 1978, edited by A. B. Schwarzkoff, W. G. Kelley, and S. B. Eliason.

The book by Gamkrelidze is an important and welcome contribution to the literature on optimal control theory.

MAGNUS R. HESTENES

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Extremal graph theory, by Béla Bollobás, London Mathematical Society Monograph No. 11, Academic Press, London, New York and San Francisco, 1978, xx + 478 pp.

Problem Solving holds an awkward place in mathematics. Everyone agrees it is fun but some question its importance. If Mathematics is the building of a castle of Theory then there is perhaps little place for the solution of an individual problem. Yet, for others, the solution of problems is far more than an amusing pastime. The creation and solution of problems determine the direction of mathematical thought. Which of these is the correct view? An easy answer is, of course, both. But the relative importance given to these not necessarily antagonistic viewpoints helps determine the nature of our subject.

Problem Solving has long played a vital role in Graph Theory. This has led to a certain subjectivity regarding the importance of any particular result. There are a myriad of possible problems and papers flood into the already overcrowded journals. Recognition of meaningful work becomes difficult but it is not impossible. With the passage of time the main currents of Graph Theory become clearly marked and the separation of the important from the mundane may begin.

Extremal graph theory is an important addition to the Graph Theory literature. There is a staggering amount of material here. Throughout, theorems are treated not as isolated results but as part of a cohesive whole.

Historical references are added to show the flow of the subject. This is what a monograph should do and it is a pleasure to see it done well. The title is somewhat misleading. As the author explains, "extremal graph theory is interpreted in a much broader sense including in its scope various structural results and any relations among the invariants of a graph especially those concerned with best possible inequalities." Topics in graph theory would have been a more accurate (but less interesting) title.

We focus on Chapter VI where Extremal Graph Theory, under the more usual narrow interpretation, is explored. In 1940, Paul Turan derived the result that is considered the forerunner of Extremal Graph Theory. Given arbitrary positive integers r, n he found the graph on n vertices with the maximal number of edges that does not contain K^{r+1}, the complete graph on $r + 1$ points. The solution is precise. Split the n vertices into r classes as evenly as possible. Let vertices be adjacent if and only if they are in different classes. This is the unique graph with the maximal number of edges. We let $t_r(n)$ denote the number of edges in this graph. The general question arose: What is the maximal number of edges a graph on n vertices can have without having a given property? Equivalently, how many edges insure that a graph on n vertices has a given property. More specifically, for any graph F let $\mathrm{ex}(n: F)$ denote the maximal number of edges in a graph with n vertices not containing F as a subgraph.

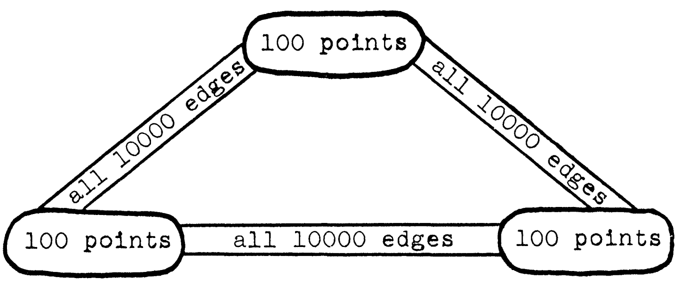

FIGURE 1. 300 points, No K^4, $t_3(300) = 3000$ edges

While Turan began the study of Extremal Graph Theory, his colleague Paul Erdös has played a decisive role. To quote from the author's preface, "The main exponent has been Paul Erdös who, through his many papers and lectures, as well as uncountably many problems, has virtually created the subject." Paul Erdös introduced the author and many others (including this reviewer) to Graph Theory.

In 1946, Erdös and Stone provided a key result. Let $K_s(t)$ denote the graph on st vertices, divided into s disjoint classes of t vertices each, with vertices adjacent iff they lie in different classes. For example, $K_s(1) = K^s$. Let r, t, $\varepsilon > 0$ be fixed and n sufficiently large. They showed that if a graph on n vertices has at least $t_r(n) + \varepsilon n^2$ edges (that is, slightly more than the number of edges that force K^{r+1}) then it contains a $K_{r+1}(t)$. The exact relationship between r, t, ε, and n and, more generally, the nature of graphs with slightly more than $t_r(n)$ edges has been the object of intense study which is well presented in this volume.

Let F be an arbitrary graph with chromatic number $\chi(F) = r + 1$. F is a subgraph of $K_{r+1}(t)$ where in the $(r + 1)$-coloring of F no color is used more than t times. From the result of Erdös and Stone

$$\mathrm{ex}(n: F) \leqslant \mathrm{ex}(n: K_{r+1}(t)) \leqslant t_r(n) + o(n^2).$$

Conversely, since F cannot be r-colored, the Turan graph with $t_r(n)$ edges described above does not contain F, providing a lower bound for $\mathrm{ex}(n: F)$. Together

$$\mathrm{ex}(n: F) = t_r(n) + o(n^2)$$

an excellent first order approximation to $\mathrm{ex}(n: F)$ when $r > 2$.

When F is bipartite ($r = 2$) the above shows only $\mathrm{ex}(n: F) = o(n^2)$ and one wishes at least to find $\alpha = \alpha(F)$ so that $\mathrm{ex}(n: F) = n^{\alpha + o(1)}$. When $F = K_2(2)$ (the 4-cycle), $\alpha = 3/2$. Essentially, a maximal graph on $n = p^2 + p + 1$ points is formed by taking the vertices of the projective plane over GF(p) and joining (a, b, c) to (x, y, z) iff $ax + by + cz = 0$. When $F = K_2(3)$, $\alpha = 5/3$. Here a graph on $n = p^3$ points is formed by taking the vertices of affine 3-space over GF(p) and joining two vertices if their squared Euclidean distance is a fixed, appropriately chosen, constant. A search for similar algebraic constructions for $F = K_2(r)$ has proven fruitless. There are, of course, many other bipartite F and the evaluation of $\alpha = \alpha(F)$ remains an open problem in all but a few cases. There is far more in this chapter but there are also seven other chapters to browse through.

Graph Theory's directions have been greatly influenced by certain key conjectures. One must mention, of course, the Four Color Conjecture (Theorem). Its influence on Graph Theory was akin to that of Fermat's Last Theorem (Conjecture) on Algebraic Number Theory. Fortunately, the fallout from this conjecture, such as studies of contradiction operations, has not ended with its resolution.

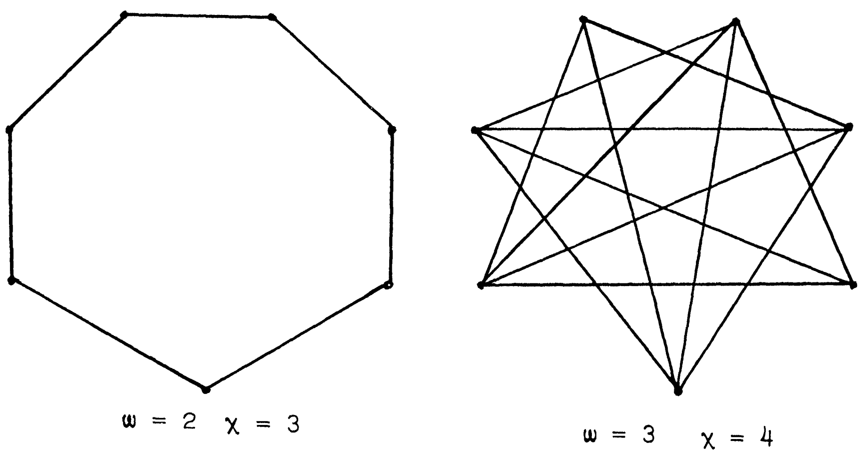

FIGURE 2. Not perfect

Berge calls a graph G *perfect* if $\omega(H) = \chi(H)$ for all subgraphs H of G. He conjectures that G is perfect if and only if it contains neither an odd cycle nor the complement of an odd cycle. This remains a most vexing problem.

The Berge Perfect Graph Conjectures are less well known to the general mathematical audience but have played an important role in modern (post 1945) Graph Theory. Two basic parameters of a graph G are the clique number $\omega(G)$ and the chromatic number $\chi(G)$. ($\omega(G)$ is the maximal clique size–a clique being a set of pairwise adjacent vertices. $\chi(G)$ is the minimal number of colors required to color the vertices of G so that adjacent vertices are colored distinctly.) Since vertices of a clique must be colored distinctly $\omega(G) \leqslant \chi(G)$. When does equality hold? Not always–the pentagon has $\omega = 2$, $\chi = 3$, as does any odd cycle. The complement of a cycle of length $2n + 1$ has $\omega = n$, $\chi = n + 1$.

Berge also made a weaker conjecture: A graph is perfect if and only if its complement is perfect. This result was proven independently by Lovasz and Fulkerson in 1972. Let $I_1, \ldots, I_m$ be a listing of the maximal independent (i.e., pairwise nonadjacent) sets of vertices on a graph G with vertex set $\{1, \ldots, v\}$. Consider the following optimization problems:

$$(*)\quad \begin{array}{l} y_1, \ldots, y_m \geqslant 0, \\ \sum_{i \in I_j} y_j \geqslant 1, \quad 1 \leqslant i \leqslant v, \\ \text{minimize } \sum_{j=1}^{v} y_j; \end{array} \qquad (**)\quad \begin{array}{l} z_1, \ldots, z_v \geqslant 0, \\ \sum_{i \in I_j} z_i \leqslant 1, \quad 1 \leqslant j \leqslant m, \\ \text{maximize } \sum_{i=1}^{v} z_i. \end{array}$$

If we allow y_j, z_i to range over the nonnegative reals, we create a pair of dual Linear Programming problems. The fundamental duality theorem of Linear Programming insures that (*), (**) have a common solution $\tau = \tau(G)$.

If we restrict y_j, z_i to be integers (and hence zero or one by the nature of the equations), we create two problems in the domain of Integer Programming. The solution to (*) is the minimal number of independent sets with union $\{1, \ldots, v\}$ which is simply $\chi(G)$. The solution to (**) is the maximal number of vertices, no two in a maximal independent set. But two vertices are in a maximal independent set exactly when they are nonadjacent so the solution to (**) is $\omega(G)$.

The inequality $\omega(G) \leqslant \tau(G) \leqslant \chi(G)$ holds for all G. The equality $\omega(G) = \tau(G) = \chi(G)$ holds precisely when both Linear Programming problems (*) and (**) have integer optima. The statement "*G* is perfect" was profitably expressed in the language of Linear and Integral Programming. These subjects are in the domain of Operations Research and are applied with great success (sometimes) in the "real world". Now the tail wags the dog and Linear and Integral Programming have yielded valuable techniques for the study of Graph Theory.

The following conjecture of Bollobás is relatively new. Let G and H be graphs on n vertices. Let G have maximal degree a and H have maximal degree b. Assume $(a + 1)(b + 1) \leqslant n$. The conjecture is that copies of G and H may be placed onto the same n vertices so that they are edge disjoint. Part of the strength of a conjecture is in its implications. Let $n = (a + 1)(b + 1)$ and let H be the union of $(a + 1)$ disjoint complete graphs on $(b + 1)$ vertices. Assuming the conjecture true: If G has maximal degree a then G may be $(a + 1)$-colored with each color used the same number of times. This equicoloration theorem has been shown by Hajnal and Szemeredi.

There are many other subjects covered in this volume and we shall not attempt to enumerate them here. The topics covered are generally discussed in depth. The book, though self-contained, would be difficult reading without some prior basic knowledge of Graph Theory. The pace is brisk and the reader is quickly brought to the frontiers of the subject.

Bollobás is a fastidious writer. The theorems are precisely stated and the proofs are carefully written. The publisher, Academic Press, has done a fine job. Most important, Bollobás is a mathematician who knows his material. In section after section he takes a set of theorems and, by appropriate concatenation plus some well chosen words of explanation, he creates a Theory.

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Multidimensional diffusion processes, by D. W. Stroock and S. R. S. Varadhan, Die Grundlehren der mathematischen Wissenschaften, vol. 233, Springer-Verlag, Berlin and New York, 1979, xii + 338 pp., $34.80.

1. *Is it best to think of a 'diffusion' as meaning* (i) *a continuous strong Markov process*, (ii) *a strong solution of an Itô stochastic differential equation, or* (iii) *a solution of a martingale problem*? Both the Markov-process approach and the Itô approach (which holds so special a place in the hearts of probabilists after the appearance of McKean's wonderful book [**7**]) have been immensely successful in diffusion theory. The Stroock-Varadhan book, developed from the historic 1969 papers by its authors, presents the martingale-problem approach as a more powerful–and, in certain regards, more intrinsic–means of studying the foundations of the subject.

The martingale-problem method has been applied with great success to other problems in Markov-process theory, both 'pure' (Stroock [**10**], . . .) and applied (Holley and Stroock [**3**], [**4**], . . .). It has conditioned our whole way of thinking about still-more-general processes (Jacod [**5**], . . .). Moreover, the method's ideas and results now feature largely in work on filtering and control (Davis [**1**], . . .).

I 'batter' you with the preceding paragraph because the authors make the uncompromising decision not 'to proselytize by intimidating the reader with myriad examples demonstrating the full scope of the techniques', but rather to persuade the reader 'with a careful treatment of just one problem to which they apply'. Halmos's doctrine 'More is less, and less is more' is thereby thoroughly tested; but if one had to choose a single totally-integrated piece of work which in depth and importance shows that probability theory has 'come of age', it would surely be the theorem towards which so much of this book is directed–or perhaps Stroock's extension of it [**10**]. Most of the main tools of stochastic-process theory are used, after first having been honed to a sharper edge than usual. But it is the formidable combination of probability theory with analysis (in the form of deep estimates from the theory of singular integrals) which is the core of the work.