and informative, it leaves the reader with a curious sense of being a spectator and not a participant.

The *LINPACK user's guide* is at the opposite extreme. At its core are 129 pages 'listing' 50 subroutines. These are written in standard Fortran and are also available on magnetic tape, hence easily executable at most computing centers. The *Guide's* main purpose is to document these programs. Written for computing professionals, it has the same sparkle as a manual explaining to expert repair mechanics the workings of an automobile engine.

Yet it is an important and meaty document, giving an authoritative picture of current mathematical software technology. Its programs have been exhaustively tested at a number of computing centers on a variety of machines, and can be certified as optimal (in the present state of the art) for solving many problems of linear algebra. Many millions of dollars of computing and programming time will be saved by using them! (The best direct and iterative methods for solving discretizations of linear elliptic boundary value problems are *not* included, however.)

Though the three books differ in most respects, they share a weakness: none of them points out the *limitations* of the methods that it explains. Thus the *LINPACK user's guide* nowhere mentions the fact that the rank of a general matrix cannot be computed in 'real arithmetic'; Nash does not comment on the possibility that his minimization algorithms fail for some functions (e.g. for $2x^2 - \pi x - 5\exp[-(10^3x)^2]$). Even Henrici, whose skillfully written programs would surely have delighted Euler or Gauss, fails to observe that the coefficients of his power series are not expressed on a computer as rational numbers, hence cannot easily be recognized as generating functions.

By making allowances for this minor weakness, the curious reader who browses through these books can acquire a realistic picture of the state of numerical mathematics today.

GARRETT BIRKHOFF

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Topologie und analysis, by Bernhelm Booss, Springer-Verlag, Berlin, Heidelberg, New York, 1977, xiv + 352 pp., $17.50.

The Atiyah-Singer index theorem; an introduction, by Patrick Shanahan, Lecture Notes in Math., vol. 638, Springer-Verlag, Berlin, Heidelberg, New York, 1978, v + 224 pp.

At the beginning of the twentieth century, mathematics had been greatly enlarged by the ideas of a number of giants, including Riemann, Cantor, Poincaré and Hilbert. Considerable effort was then expended on understanding and developing the whole new areas of mathematics which had been created. And this plus the general trend toward axiomatization meant that a parochial view largely dominated mathematics. In the last couple of decades, however, that has changed and some of the most exciting developments have come from the interaction, often in unexpected ways, of different parts of mathematics. Perhaps the most striking example of this and certainly one of the most important results of the last few decades is the index theorem of Atiyah and Singer.

In the early study of concrete operators by Fredholm, Hilbert, Riesz and others at the beginning of this century, it was shown that an operator had only the zero vector in its null space if and only if the operator was onto (which is the so-called Fredholm alternative). This resulted from the limited class of operators being investigated (for example, integral operators defined by a square integrable kernel), and indeed, when integral operators with singular kernels were allowed, this was no longer the case. F. Noether and others, however, found that the difference between the dimensions of the null space and the co-null space (or the orthogonal complement of the range), which before had been zero, was still expressible in terms of topological invariants of the data used to define the operator. For example, Kreĭn showed for Wiener-Hopf operators that the index is minus the winding number of the Fourier transform of the kernel function which defines it. Historically, operators defined by singular kernels of Cauchy type were considered earlier by F. Noether, Plemelj, Carleman and Giraud on the circle and line and eventually by Calderón-Zygmund in $\mathbf{R}^n$. Gradually the abstract class of Fredholm operators was identified as those with closed range and finite dimensional null space and co-null space and the analytical index of such operators was defined to be the difference of these dimensions. Fredholm theory was largely codified for operators on Hilbert space (and extended to operators on Banach spaces) in a classic paper of Gohberg and Kreĭn. Using a result of Atkinson the Fredholm operators on a complex Hilbert space were shown to be precisely those which are invertible modulo the ideal of compact operators. Recall that an operator on a complex Hilbert space is compact if it is the norm-limit of finite rank operators. Hence, a compact operator on an infinite dimensional space can be viewed as negligible and therefore an operator is Fredholm if and only if it is almost invertible. Moreover, index was shown to be a continuous homomorphism to $\mathbf{Z}$ and since $\mathbf{Z}$ is discrete it follows that index is constant on components. In fact, for operators on a complex Hilbert space index determines the components and thus is the only perturbation invariant for the set of Fredholm operators.

Although the study of partial differential equations is almost as old as analysis, an operator theoretic point of view is of relatively recent origin. The early study of partial differential equations arose from physics and most effort was directed toward understanding the solution of the equations of physics. Even so, by this century the consideration of partial differential equations defined on manifolds was unavoidable. This resulted not only from the needs of physics but also from the study of such areas of mathematics as global differential geometry. In particular, Hodge theory can be viewed as the study of Laplace's equation on a manifold. And based on results of Weyl and many other mathematicians it was realized that a large class of operators defined by partial differential equations on compact manifolds were in fact Fredholm operators and hence had an index.

A differential operator acts naturally on the space of C^∞ functions but can be extended naturally to define a bounded operator between Sobolev spaces of the appropriate order. Basically the Sobolev space $H^k(M)$ on a manifold M is defined by considering functions which possess distributional derivatives in L^2 up to order k and is endowed with a norm which takes into account the usual L^2-norm of all these derivatives. Then $H^k(M)$ is a Hilbert space. If P is a differential operator of order m defined on M, then P defines a bounded operator from $H^k(M)$ to $H^{k-m}(M)$. Moreover, it follows from a result of Rellich-Sobolev-Kondrashov that a differential operator of order less than m defines a compact operator from $H^k(M)$ to $H^{k-m}(M)$. Hence ignoring the lower order terms of P only amounts to a compact perturbation of the orginal operator. Therefore we consider the homogeneous part P_m of P consisting of the terms of order m and the action of P_m at a point can be approximated on the tangent space by the Fourier transform of a multiplier on the cotangent space. This function is called the principal symbol σ_P of P. If $M = \mathbf{R}^n$ and P has constant coefficients, then σ_P is just the polynomial obtained by replacing $i\partial/\partial x_j$ by ξ_j. More generally, if E and F are smooth vector bundles over M, then differential operators acting between the C^∞ cross-sections can be defined and in general, the principal symbol σ_P of such an operator P is a cross-section of the bundle $\operatorname{Hom}(\tilde{E}, \tilde{F})$ over the cotangent bundle $T^*(M)$, where $\tilde{E}$ and $\tilde{F}$ denote the pullbacks of E and F over M to $T^*(M)$. The differential operator P is said to be elliptic if $\sigma_P(x, \xi)$ is an isomorphism for each (x, ξ) in $T^*(M)$, $\xi \neq 0$.

If M is a compact manifold without boundary, then the operator P is Fredholm if and only if it is elliptic. Since the lower order terms of P define a compact operator, the symbol determines T up to a compact perturbation and since index is continuous, it follows that only the homotopy type of σ_P is important. Thus we arrive at the problem of calculating the index of an elliptic differential operator in terms of topological invariants of its symbol. This is essentially the problem raised by Gelfand in his seminar in the late fifties, although he considered specifically elliptic boundary-value problems. Early results on this were obtained by Agranowich, Dynin and Volpert. Seeley also obtained results for the index of singular integral operators on Euclidean space generalizing those of Michlin and Gohberg. Then in 1963 Atiyah and Singer formulated and proved a formula for the index of elliptic differential operators on compact manifolds. Their motivation included the idea of giving an alternate proof of the Hirzebruch signature theorem by expressing signature as the index of a specific operator and then calculating the index in terms of topological invariants. In addition, they used Hirzebruch's proof of his Riemann-Roch theorem as a model for the proof of their index theorem. Although only an outline of the proof was given in their announcement, complete proofs were quickly provided in the published seminar notes from the Palais seminar at the Institute for Advanced Study in Princeton and the Cartan-Schwartz seminar in Paris the following year.

The first proof of Atiyah and Singer involved mostly ordinary cohomology theory, characteristic classes, cobordism and integro-differential operators. Their index formula involves the Todd class of the tangent bundle of the manifold which is a rational cohomology class and the Chern character of the bundle defined on the Thom space using the symbol σ_P restricted to the cosphere bundle as a clutching function. Using deep results from cobordism theory, eventually one is able to reduce the proof of the general index formula to that of calculating the index of a generalized signature operator.

In the course of this proof two topics were touched upon which have undergone tremendous development in the intervening years. Although both had been introduced earlier, their rapid dissemination and development were spurred, at least in part, by their connection with the index theorem. Following Grothendieck's ideas in algebraic geometry, Atiyah and Hirzebruch studied the ring completion $K(X)$ of the collection of vector bundles on a space X under Whitney sum and tensor product and showed its importance in many problems in topology. The periodicity theorem of Bott was reinterpreted to form the cornerstone of K-theory showing that a generalized cohomology theory is obtained. Since the index theorem provides a homomorphism from the bundles on the Thom space defined by the symbols of differential operators to $\mathbf{Z}$, it can be factored through the associated K-group. Now not all elements of this group arise as the bundle defined by the symbol of a differential operator. However, there is a generalization called pseudo-differential operator due to Kohn-Nirenberg and Hörmander which possesses many of the same properties as differential operators, including a symbol, which corrects this. And when this symbol is invertible, the operator is Fredholm and hence possesses an index. Since the bundles defined by the symbols of pseudo-differential operators generate the K-group we get a homomorphism to the integers called the analytical index. By the time Atiyah and Singer gave a detailed exposition of their work, they had obtained a new expression for the index formula in terms of K-theory and a new proof. For the formula they defined topologically another homomorphism from the K-group to $\mathbf{Z}$ called the topological index and they then proceeded to show that the two indices were equal. Their proof was modeled on Grothendieck's proof of his generalization of the Hirzebruch-Riemann-Roch theorem. A key step in the proof involved embedding the general case in Euclidean space and thereby reducing the proof to calculating the index of classical differential operators defined on open subsets of Euclidean space. The cohomological version of the formula can be obtained as an exercise in algebraic topology relating K-theory to ordinary cohomology theory.

There were several other developments in the sixties which we should mention. First, the index theorem was extended to operators which commute with a compact Lie group. The index in this case is an element of the representation ring of the group and equivariant K-theory is the relevant tool for the proof. This latter notion was developed largely by Atiyah and Segal with this application in mind. Second, an index theorem for families was also proved in which the operator depends continuously on a parameter in a compact space. At each point one has a Fredholm operator and using the fact proved by Atiyah and Janich that the Fredholm operators on complex Hilbert space are a classifying space for K-theory, an analytical index is obtained which is an element of the K-group of the parameter space. A topological description of this element completes the index formula. Further, various fixed point theorems in this context were proved by Atiyah, Bott and Singer. Finally, many applications of this work were obtained especially to topology and geometry.

In the early seventies a new approach to the index theorem was found by Atiyah, Patodi, and Singer based on a study of the heat equaton on manifolds. This involved considerably more analysis than earlier proofs and, in addition, local differential geometry. It increased, however, the number of areas that the index theorem makes contact with. Finally, Fedosov has carried this development further to obtain a proof based almost entirely on operator theory and the theory of pseudo-differential operators. It involves the direct construction of the Fredholm inverse by writing out its total symbol.

A different development can be traced to an attempt of Atiyah at giving a concrete realizaton of the homology theory dual to K-theory. He introduced the notion of a generalized elliptic operator and showed that they could be used to form the cycles for the K-homology groups. What the particular equivalence was, was left open and was resolved independently by Kasparov and Brown, Fillmore and this reviewer. In the latter approach concrete elements for K-homology were shown to be provided by C^*-algebra extensions and important relations with operator theory were established. In Kasparov's approach connections with rather important problems in algebraic topology were a central consideration. In both cases a version of the index theorem is apparent in which an elliptic pseudo-differential operator is used to define directly an element of the K-homology group. This is the "analytical index" and the index theorem consists of describing that element topologically. Further there is an extension of the index theorem to a class of singular spaces by Baum, Fulton, and McPherson. Their methods are based on the Grothendieck proof. Finally, Connes has extended the index theorem to differential operators on a foliation using heat equation methods. The operators in this instance are Fredholm relative to a II^{∞}-factor and have a real valued index.

The theory surrounding the index theorem has had important repercussions in much of mathematics in addition to being a very important result in its own right. Moreover, many applications have been made and the story is far from over.

How does one go about learning about the index theorem? Although the original papers are clear and incisive and a must for any serious student, the prerequisites for reading them are considerable. The two books under review offer different approaches to the index theorem.

In the book of Booss all the prerequisites are presented in considerable detail including many heuristic remarks. This means Booss develops Fredholm theory, pseudo-differential operators, and algebraic topology starting at a level more or less appropriate to an advanced graduate student. I can recommend this book to advanced topics courses in which the students and instructor might feel obliged to consult other sources along the way. It might also be satisfactory for good students to study on their own. Only the embedding proof of the index theorem is presented in detail although other proofs are outlined. To complete such a course of study, the original papers should then be read. I believe the book would be quite satisfactory for this.

The book of Shanahan is directed more at mathematicians who want to know something about the index theorem and its applications. The classical examples named after de Rham, Dolbeault, Hodge and Dirac are presented in considerable detail. The embedding proof of the index theorem is outlined along with that of the equivariant index theorem and the fixed point theorem. Various applications of these results are also presented. The book is quite successful in doing what it attempts.

While the index theorem has not yet made it into graduate texts, these two books are a good beginning and given the ongoing importance of the index theorem should be useful to those wanting to learn about it.

RONALD G. DOUGLAS

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Probabilities and potential, by Claude Dellacherie and Paul-André Meyer, Mathematics Studies, Volume 29, North-Holland Publishing Company, Amsterdam and New York, 1978, xii + 190 pp., $29.00.

In the preface of his 1953 book [3], J. L. Doob wrote "Probability is simply a branch of measure theory, with its own special emphasis and field of application .... Using various ingenious devices, one can drop the interpretation of sample sequences and functions as ordinary sequences and functions, and treat probability theory as the study of systems of distribution functions. · · · such a treatment · · · results in a spurious simplification of some parts of the subject, and a genuine distortion of all of it." I believe that today the vast majority of probabilists would agree with Doob's statement of more than a quarter of a century ago. The thought of trying to state, let alone explain, the strong law of large numbers, for example, without using sample sequences seems ludicrous. The mathematical model commonly accepted today for treating sample sequences and functions is measure theory via the Kolmogorov axioms. As long as one deals with sequences most probabilists are happy with the measure theoretic foundations of the subject. However, this sense of contentment is rapidly dissipated when treating sample functions; that is, uncountable families of random variables. This is because, until quite recently, most probabilists were uncomfortable with the type of measure theory that is required to discuss sample functions.

The study of sample functions of a stochastic process has a long and varied history. In [10], Loève has emphasized that Lévy always thought in terms of sample paths and that this approach led to his beautiful results beginning in the middle 1930's on the structure of additive processes, the fine structure of Brownian paths, and the bizarre (at the time they were published in 1951) possibilities for the sample paths of a continuous parameter Markov chain. In spite of Wiener's construction of Brownian motion in the 1920's, there was hardly any theory of continuous parameter stochastic processes in 1935. Beginning about 1936 and culminating in his 1953 book, Doob developed a rigorous foundation for treating such questions. At about the same time Doob and later Snell were establishing the sample function properties of martingales and submartingales which were to be fundamental for later developments. These results were given a definitive treatment in the 1953