examples named after de Rham, Dolbeault, Hodge and Dirac are presented in considerable detail. The embedding proof of the index theorem is outlined along with that of the equivariant index theorem and the fixed point theorem. Various applications of these results are also presented. The book is quite successful in doing what it attempts.

While the index theorem has not yet made it into graduate texts, these two books are a good beginning and given the ongoing importance of the index theorem should be useful to those wanting to learn about it.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 3, May 1980

0002-9904/80/0000-0222/$02.00

*Probabilities and potential*, by Claude Dellacherie and Paul-André Meyer, Mathematics Studies, Volume 29, North-Holland Publishing Company, Amsterdam and New York, 1978, xii + 190 pp., $29.00.

In the preface of his 1953 book [3], J. L. Doob wrote "Probability is simply a branch of measure theory, with its own special emphasis and field of application .... Using various ingenious devices, one can drop the interpretation of sample sequences and functions as ordinary sequences and functions, and treat probability theory as the study of systems of distribution functions. · · · such a treatment · · · results in a spurious simplification of some parts of the subject, and a genuine distortion of all of it." I believe that today the vast majority of probabilists would agree with Doob's statement of more than a quarter of a century ago. The thought of trying to state, let alone explain, the strong law of large numbers, for example, without using sample sequences seems ludicrous. The mathematical model commonly accepted today for treating sample sequences and functions is measure theory via the Kolmogorov axioms. As long as one deals with sequences most probabilists are happy with the measure theoretic foundations of the subject. However, this sense of contentment is rapidly dissipated when treating sample functions; that is, uncountable families of random variables. This is because, until quite recently, most probabilists were uncomfortable with the type of measure theory that is required to discuss sample functions.

The study of sample functions of a stochastic process has a long and varied history. In [**10**], Loève has emphasized that Lévy always thought in terms of sample paths and that this approach led to his beautiful results beginning in the middle 1930's on the structure of additive processes, the fine structure of Brownian paths, and the bizarre (at the time they were published in 1951) possibilities for the sample paths of a continuous parameter Markov chain. In spite of Wiener's construction of Brownian motion in the 1920's, there was hardly any theory of continuous parameter stochastic processes in 1935. Beginning about 1936 and culminating in his 1953 book, Doob developed a rigorous foundation for treating such questions. At about the same time Doob and later Snell were establishing the sample function properties of martingales and submartingales which were to be fundamental for later devclopments. These results were given a definitive treatment in the 1953 book. All of this required a serious use of measure theory and, consequently, Doob's was a lonely voice in the wilderness during much of this time. I believe it is fair to say that most probabilists in the late 1930's and the 1940's (and, perhaps, even more recently) were distressed by the type of measure theory used in sample function analysis. Of course, there were exceptions that proved the rule–an outstanding example being *K*. Ito who in 1942 give a rigorous treatment of Lévy's theory of additive processes and in 1951 constructed a wide class of Markov processes directly in terms of their sample paths.

The following sheds some light on the state of the art in 1940. In the first paragraph of his paper [6], Feller gives an informal definition of a Markov process which is essentially the modern one. But he immediately qualifies it with the footnote: "This is, of course, not meant to be a strict definition; as a matter of fact, we shall be concerned only with the function $P(\tau, x; t, \Lambda)$ which will be defined purely analytically." Here $P(\tau, x; t, \Lambda)$ is the transition function of $X$; that is, it is the conditional probability that $X(t) \in \Lambda$ given $X(\tau) = x$ with $\tau < t$. At the top of the next page, Feller writes down the Chapman-Kolmogorov equations for $P$ and then formally defines a Markov process as a process whose transition function satisfies these equations. Such definitions may be traced back through Feller's 1936 paper [5] to Kolmogorov's famous paper [9] in 1931. The point is that during this era, rigorous definitions were rarely attempted, and it was even rarer that they were taken seriously.

This situation began to change rapidly in the 1950's, in no small part due to the appearance of Doob's book in 1953. At the same time a number of papers were appearing, mainly in the United States and the Soviet Union, dealing with sample function properties of Markov processes and the strong Markov property. This last property arose quite naturally when the process is considered at random times which depend on the particular sample function such as the first time a process enters a given set. Special mention should be made of Chung's deep results on continuous parameter Markov chains during this period. Also Blumenthal, Doob, Hunt, Kinney, McKean, and Ray in the United States and Dynkin and Jushekevitch in the Soviet Union, among others, made important contributions during the period 1951–1956.

Then in 1957–1958 Hunt's memoir [7] appeared. This was a monumental work of nearly 170 pages that contained an enormous amount of truly original mathematics. In retrospect it seems to me that probabilists were rather slow to appreciate what Hunt had accomplished. (On the other hand potential theorists were immediately struck by §15 in which Hunt showed the *identity* of a form of potential theory and the theory of Markov processes.) Undoubtedly this was because Hunt took measure theory seriously and this put off the great majority of probabilists even then. In addition, it must be stated that it was not easy reading.

The most relevant part of Hunt's memoir for the present review is §2 in which he introduced Choquet capacity theory as a tool in probability theory. He used Choquet's theorem on the capacitability of analytic sets to show that the entrance time $D_A$ of an analytic set $A$ is a random variable (i.e. measurable) and more importantly that $D_A$ could be approximated by the entrance times of compact subsets. Classically, even for Brownian motion, one could only treat $F_\sigma$ sets $A$, although before Hunt, Doob in [**4**] had proved the result in the Brownian case for capacitable sets using results of Cartan in classical potential theory.

The mere recital of these facts masks what is involved since the result is *false* if one interprets measurability in the most naive way. To be precise consider the following situation. Let $E$ be a complete separable metric space and let $\Omega$ be the collection of all right continuous functions $\omega$: $\mathbf{R}^+ \to E$ which have left limits at each $t > 0$. Here $\mathbf{R}^+$ stands for the nonnegative reals. For each $t \geqslant 0$ let $X_t$: $\Omega \to E$ be the coordinate map $X_t(\omega) = \omega(t)$. The natural $\sigma$-algebra $\mathcal{F}^0$ on $\Omega$ is generated by these coordinate maps, that is $\mathcal{F}^0 = \sigma\{X_t: t \geqslant 0\}$. If $A \subset E$, then the entrance time of $A$ is defined by

$$D_A(\omega) = \inf\{t \geqslant 0: X_t(\omega) \in A\}$$

where the infimum of the empty set is zero. If $A$ is open, the right continuity of $t \to X_t(\omega)$ trivially implies that $\{D_A < t\} \in \mathcal{F}^0$. (More precisely, $\{D_A < t\} \in \mathcal{F}_t^0 = \sigma\{X_S: s \leqslant t\}$ so that $D_A$ is a "stopping time".) Recently Dellacherie [**2**] has shown that if $A$ is closed and $D_A$ is $\mathcal{F}^0$ measurable, then $A$ is necessarily open! So far this is pure set theory–no probability is involved. What is true is that if $P$ is *any* probability measure on $(\Omega, \mathcal{F}^0)$ and $A$ is Borel (or even analytic) in $E$, then $D_A$ is measurable relative to the completion of $\mathcal{F}^0$ under $P$. In fact for each $t$, $\{D_A < t\}$ is "analytic" over $(\Omega, \mathcal{F}_t^0)$. Hunt himself did not prove results quite as general as this about $D_A$–the actual results described above are due to Meyer.

Beginning about 1960 a very rapid development of probabilistic potential along the lines of Hunt got under way. Of course, there were other important ideas being incorporated into the subject. Special mention should be made of the work of Kac. In particular his 1951 paper [**8**] was enormously influential. Also flowing into this stream were the ideas of Doob (conditional Brownian motion and boundary theory), Ito (stochastic calculus and (with McKean) diffusions), and Dynkin and his school (generators, additive functionals, diffusions), among others. At first the probabilistic side was concerned mainly with Markov processes, but after Meyer's work on the Doob decomposition of supermartingales in 1962–1963, more and more attention was directed to martingales. Of course, the martingale and Markovian aspect did not develop independently of each other, but rather as complementary facets of a single theory. For example, Meyer's proof of the Doob decomposition was a generalization of earlier work of Volkonski, Šur, and Meyer, himself, in the theory of Markov processes. During the 1960's and into the 1970's a large arsenal of tools was developed, mainly by Meyer and those working under his influence, for attacking various problems concerning Markov processes, martingales, stochastic integrals, and the like. This body of knowledge has come to be known as the "general theory of processes" or sometimes as "Strasbourg probability".

The book under review is the first volume in a projected multi-volume exposition of probability and potential theory. As the authors point out in their preface, the present volume contains no potential theory and only the foundations of stochastic processes in probability. (The manuscript of a second volume dealing with martingale theory, but not stochastic integrals, is finished and should appear shortly, at least in French.) This first volume contains four chapters. The first two are short chapters in which the authors collect some facts from measure theory and introduce the basic definitions of probability theory. The first two parts of Chapter III discuss analytic sets and capacity theory. The treatment is much closer to that in the first edition of this book **[11]** than to that in Dellacherie's book **[1]**. The third part treats bounded Radon measures and includes the theorems of Kolmogorov and Prokhorov and a discussion of narrow (weak) convergence. The first two parts of Chapter IV contain the foundations of the theory of continuous parameter stochastic processes. The remainder of Chapter IV begins the discussion of the general theory of processes and carries it about as far as one can go without using martingales. Thus it contains the basic definitions of optional and predictable times and $\sigma$-algebras, the measurability of debuts, and the section theorems. The projection theorems which require martingales are deferred to Volume II. The treatment here differs from that in Dellacherie **[1]** in that the "usual hypotheses" are not invoked as a blanket assumption from the beginning. As the authors point out this is not done for the sake of generality, but because in many applications one must deal with several (uncountably many) probabilities simultaneously in which case the usual hypotheses, involving completeness, are inappropriate. Finally there are very extensive indices of terminology and notation which greatly enhance the use of this book as a reference.

The general theory of processes is really the study of a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Here $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t)_{t>0}$ is an increasing family of sub-$\sigma$-algebras of $\mathcal{F}$. Perhaps it is surprising that one can develop so much deep, beautiful, and important probability in such a general framework, and that such a theory should have important applications to many areas of probability such as Markov processes, martingale theory, stochastic differential equations, and filtering theory. For example in my own field of Markov processes, general theory has been used to attack and solve many problems that seemed, to me at least, unapproachable fifteen years ago. Having said all this, perhaps I should emphasize the obvious: it is also true that much of modern probability falls outside of the scope of this theory.

The general theory of processes has become a somewhat controversial subject. Some probabilists tend to denigrate it as too abstract, arid, or lacking in probabilistic content. Although I can sympathize with those who find the measure and set theory distasteful, I believe that it has proved itself to be an important part of probability. Only time will determine the ultimate importance of this theory which, at the moment, is enjoying a period of vigorous growth. In the meanwhile the current generation of probabilists can only be grateful to Dellacherie and Meyer for their thorough exposition and hope that they do not tire in the task that they have set themselves.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Number 3, May 1980

0002-9904/80/0000-0223/$03.25

*Ordinary differential equations*, by V. I. Arnold, translated from the Russian by Richard A. Silverman, MIT Press, Cambridge, Massachusetts, 1978, x + 280 pp., $8.95.

The past twenty years have witnessed a revolution in the field of ordinary differential equations. It is not uncommon to attend a seminar on differential equations and not even hear the words differential equations, let alone see one written on the board. The "in phrase" these days is dynamical systems, and the language spoken is often the language of topology and differential geometry. *Ordinary differential equations* by the famed Soviet mathematician V. I. Arnold is a superb introduction to the modern theory of differential equations, and while reviewing this book, it is instructive to take a closer look at the profound changes that have occurred in this field. We begin with the fundamental concept of a dynamical system.

Consider a system that is evolving in time. Let $x$ denote the initial state of the system, and $g^t x$ its state at time $t$, with $g^0 x = x$. The set $M$ of all possible states is called the phase space of the system, and the individual states $x$ are called phase points. Suppose, moreover, that the mappings $g^t$ satisfy the group property

$$g^{t+s}x = g^t(g^s x) \tag{1}$$

and that $g^t$ and $(g^t)^{-1}$ satisfy appropriate continuity conditions. The set of mappings $g^t$, together with the phase space $M$ is called a dynamical system.

Dynamical systems occur very naturally in the study of ordinary differential equations. Let

$$\dot{x} = v(x) \tag{2}$$

be a differential equation defined on a domain $M$ of $n$ dimensional Euclidean