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Metric planes and metric vector spaces, by Rolf Lingenberg, John Wiley & Sons, New York, 1979, xiv + 209 pp.

Given a geometry there are canonically associated groups. For example, geometries arising from bilinear forms yield the symplectic and orthogonal groups and, in the latter case, important subgroups such as the rotation group, the spinorial kernel and the commutator subgroup. In both cases appropriate factor groups (the projective groups) produce families of simple groups. These classical groups have held a prominent position in 20th century mathematics and the algebraic aspects are treated in the classic works of Artin, Dieudonné, Eichler, and O'Meara (**[A]**, **[D1]**, **[D2]**, **[E]**, **[O'M]**). Of primary importance are the investigations on generators, structure, isomorphisms and automorphisms.

It is natural to pose the inverse problem–given a group G of a certain type is there a geometry associated to G and what is its character. This is part of the investigation of Tits in **[T]** who reminds us in the introduction of the construction of a complex projective space from $SL_n(\mathbf{C})$ by using the maximal parabolic subgroups as subspaces; incidence of subspaces occurring when the intersection of two maximal parabolics is parabolic. Along a slightly different line are the works of Bachman, Lingenberg, Sperner, and Strubecker, and the book under review.

A fundamental object and object of study in *Metric planes and metric vector spaces* is that of an *S-group*. Indeed, 5 of the 8 chapters and approximately 3/4 of the pages are devoted to this topic. This concentration will be reflected in the review.

An S group is a pair (G, S) consisting of a group G generated by a subset S of the full set J of involutions subject to the following axiom

Axiom S

$$a \neq b \quad \textit{and} \quad abx, aby, abz \in J \quad \textit{implies} \quad xyz \in S.$$

The relation κ defined by

$$(a, b, c) \in \kappa \Leftrightarrow abc \in J \tag{$*$}$$

is a *ternary equivalence relation* on the set S, meaning that κ is a subset of $S \times S \times S$ satisfying:

(E1) (*Reflexivity*). *If a, b, and c are not mutually distinct, then* $(a, b, c) \in \kappa$.

(E2) (*Symmetry*). *If* $(a, b, c) \in \kappa$ *and* π *is a permutation of* $\{a, b, c\}$ *then* $(\pi(a), \pi(b), \pi(c)) \in \kappa$.

(E3) (*Transitivity*). $a \neq b$ *and* $(a, b, c) \in \kappa$ *and* $(a, b, d) \in \kappa$ *imply that* $(a, c, d) \in \kappa$.

Axiom S is needed to verify (E3) only.

If S contains at least two elements the pair (S, κ) is called an *incidence structure*; the elements of S are called *lines* and are denoted by lower case latin letters. Three lines a, b, c are *concurrent* if and only if $(a, b, c) \in \kappa$. The subsets $S(a, b) = \{x | (a, b, x) \in \kappa\}$ are called *points* and are denoted by capital letters. Thus a point is a collection of lines all of which are concurrent with two given lines. A line a is said to be *incident* with a point B if and only if $a \in B$. A basic lemma, proved for general incidence structures and not just for S groups, then yields the following familiar incidence relations:

$a, b \in A, B$ implies $a = b$ or $A = B$.

Three lines a, b, c are concurrent if and only if there is a point A incident with each of a, b, and c.

For every point A, there are at least two lines incident with A.

In this context the incidence structure $(S, \kappa(G, S))$ is called the *group plane* of the S-group.

More generally, and this is the author's initial point of departure, an *incidence structure* is a pair (L, κ) consisting of a set L having at least two elements (the lines) and a ternary equivalence relation κ on L i.e. κ is a subset of $L \times L \times L$ satisfying (E1), (E2), and (E3). The points are the subsets $L(a, b) = \{x | (a, b, x) \in \kappa\}$; concurrency of lines and incidence of points and lines are defined as for S groups.

As an example we consider a three-dimensional metric vector space (V, Q) defined over a field K. Here Q is a quadratic form on the vector space V and Q is not identically zero on V. The lines $L = L(V, Q)$ are the anisotropic lines of V and the ternary equivalence relation $\kappa = \kappa(V, Q)$ is given by

$$\kappa(V, Q) = \{(\langle A \rangle, \langle B \rangle, \langle C \rangle) | Q(A), Q(B), Q(C) \neq 0 \text{ and } A, B, C \text{ are linearly dependent}\}.$$

This is a fundamental example of an incidence structure that is Δ-*connected,* a concept of crucial importance in the book; to define this term we return to the general setting of an incidence structure (L, κ). Two points A and B are said to be *connected* (by a line) if $A \cap B \neq \varnothing$. The point A is Δ-*connected* (dreiseitverbindbar) if A is connected with at least one point of any triple B, C, D of distinct, pairwise connected points. The point A is 1-Δ-*connected* if it is Δ-connected and if there exists at least one such triple of points B, C, D for which A is connected with exactly one point of the triple. We say that A is 3-Δ-*connected,* or *completely connected,* if A is connected with all points. A Δ-connected point that is neither completely connected nor 1-Δ-connected is called a 2-Δ-*connected point*. An incidence structure (L, κ) is called Δ-*connected* if every point of (L, κ) is Δ-connected.

One of the main theorems in the book renders equivalent *complete S-group planes* and the incidence structures of three-dimensional metric vector spaces; an S-group plane is complete if it is Δ-connected and contains a quadrilateral (i.e. four lines no three of which are concurrent) or if S contains exactly two lines or four lines, three of these concurrent and orthogonal to the fourth. The theorem proved is:

MAIN THEOREM 6.1. *Every complete S-group plane is isomorphic to the incidence structure over a suitable three-dimensional metric vector space* (V, Q) *such that* $\dim V^{\perp} \leqslant 2$.

Conversely, any incidence structure $I(V, Q)$ *over a three-dimensional metric vector space* (V, Q) *such that* $\dim V^{\perp} \leqslant 2$ *is isomorphic to a suitable complete S-group plane.*

In the preface the author writes:

This book is devoted to a certain domain of plane geometry for which both *incidence and metric concepts, such as orthogonality or reflections, are defined.*

The theory developed can be regarded as

(a) *a purely geometric theory based on the concept of incidence structures with orthogonality or with reflections, mainly as a treatment of Euclidean and non-Euclidean planes and certain subplanes of these planes,*

(b) *a theory of three-dimensional metric vector spaces with their natural geometric interpretation,*

(c) *a theory of special types of S-groups and their group planes.*

These areas of plane geometry, (a) *to* (c), *are only different representations of the same theory, and one of the most important tasks of the book is to verify these relations.*

The theorem cited above is the equivalence of (b) and (c).

As an example of the nexus between (a) and (c) we cite the S-group theoretic versions of 2 classical geometric theorems; the theorem on the altitudes of a triangle and the theorem of Desargues.

In the group plane of an S-group (G, S) there is a natural notion of orthogonality given by the binary relation ω

$$(a, b) \in \omega \Leftrightarrow ab \in J.$$

This relation is irreflexive and symmetric and brings to mind the case of permuting symmetries in orthogonal groups. The theorem on the altitudes of a triangle then takes the following form:

Let $(abc)^2 \neq 1$ *and* $au, bv, cw \in J$; $bcu, cav, abw \in J$. *Then* $uvw \in J$.
Thus the assumptions, stated geometrically, are that a, b and c are noncollinear lines, a and u, b and v, and c and w are orthogonal lines and b, c, u; c, a, v and a, b, w are collinear. The conclusion is that the altitudes u, v, and w are collinear.

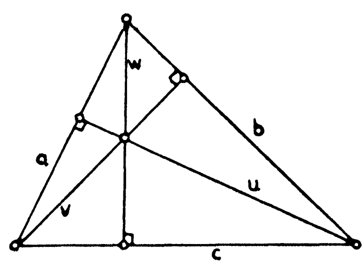

FIGURE 1

Desargues' theorem has the following form:

Suppose we have the following configuration of points and lines in the S-group plane of (G, S): g_i, a_i *for* $i = 1, 2, 3$ *and* o, b_1, b_2 *are lines*; O, P_i, Q_i, A_i *for* $i = 1, 2, 3$ *are points*; *and the following incidences hold*: $g_i \in O, P_i, Q_i$; $g_k \notin P_i, Q_i$; $o \in A_i$; $P_i \neq Q_k$ *for* $i, k = 1, 2, 3$ *and* $i \neq k$. *Also* $a_i \in A_i, P_k, P_1$ *for any cyclic permutation* $i, k, 1$ *of* $1, 2, 3$. *Also* $b_1 \in A_1, Q_2, Q_3$; $b_2 \in A_2, Q_1, Q_3$; $o \notin O$.

The points P_1, P_2, P_3 *and the points* $Q_1\ Q_2, Q_3$ *are not collinear. Assume that the points* O, A_1, A_3 *are completely connected. Then there exists a line* b_3 *such that* $b_3 \in A_3, Q_1, Q_2$.

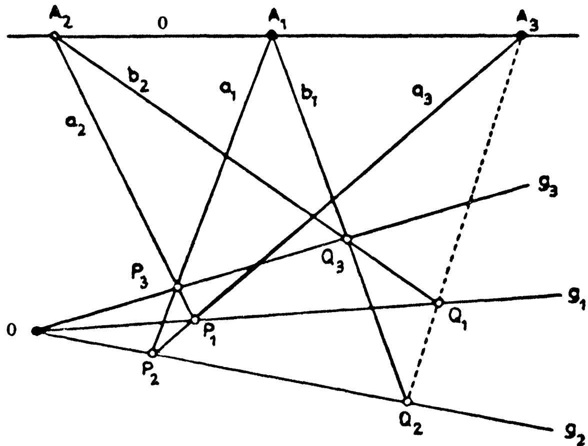

FIGURE 2

Now return to the setting of metric vector spaces. The *radical* of (V, Q), denoted $\mathrm{rad}(V, Q)$, is defined in the usual fashion; it is an isotropic subspace of V i.e. $Q(x) = 0$ for all x in $\mathrm{rad}(V, Q)$. Several authors use the term totally isotropic for such a subspace. If T is an isotropic subspace of maximal dimension the number $\dim T - \dim \mathrm{rad}(V, Q)$ is well defined and called the *Witt index* of (V, Q); denoted $\mathrm{ind}(V, Q)$. Up to isometry there are 5 distinct types of three-dimensional metric vector spaces; each type determined by the values of $\mathrm{ind}(V, Q)$ and $\dim \mathrm{rad}(V, Q)$. The respective incidence structure $I(V, Q) = (L(V, Q), (V, Q))$ is called

	$\mathrm{ind}(V, Q)$	$\dim \mathrm{rad}(V, Q)$
an elliptic coordinate plane, if	0	0
a Euclidean coordinate plane, if	0	1
a Strubecker coordinate plane, if	0	2
a hyperbolic-metric coordinate plane, if	1	0
a Minkowskian coordinate plane, if	1	1

Synthetic definitions of these 5 planes are provided within the context of complete S-group planes by means of the following axioms:

AXIOM H. *There exist two distinct* 2-Δ-*connected points which are connected by a line.*

AXIOM EM. *There exist two* 2-Δ-*connected points that are not connected by a line.*

AXIOM $\Delta 1$. *There exists a* 1-Δ-*connected point.*

The converses of these three axioms are denoted by $\neg H$, $\neg EM$, and $\neg \Delta 1$, respectively.

For the sake of brevity we consider in the sequel only the complete S-group planes that are Δ-connected and contain a quadrilateral. With this convention a complete S-group plane is called:

an elliptic plane, if	$(\neg H)$, $\neg EM$, $\neg \Delta 1$ *holds*
a Euclidean plane, if	$\neg H$, EM, $\neg \Delta 1$ *holds*
a Strubecker plane, if	H, EM, $(\neg \Delta 1)$ *holds*

a hyperbolic-metric plane, if H, $(\neg EM)$, $\Delta 1$ *holds*

a Minkowskian plane, if $(\neg H)$, EM, $\Delta 1$ *holds.*

(On the right assumptions that are bracketed follow from the ones not in brackets.)

In a long series of lemmas, propositions, and theorems, the metric and synthetic versions of these five planes are shown to be equivalent and numerous other equivalences are established. To taste the flavor of a typical theorem we must consider *projective incidence structures* and *substructures* and *Pappian* or *Desarguesian projective incidence structures.*

Let (P, L, I) be a projective plane, i.e. two disjoint sets P, the points, and L, the lines, and an incidence relation $I \subset P \times L$ satisfying:

(P1) *Given any pair of points there exists a line incident with both. Given any pair of lines there exists a point incident with both.*

(P2) *If both of the points* A, B *are incident with both of the lines* c, d, *then* $A = B$ *or* $c = d$.

(P3) *There exist four points no three of them incident with the same line.*

Then three lines a, b, c in L are concurrent in the projective plane if there is a point A in P such that A is incident with a, b, and c.

Set $\kappa = \{(a, b, c) | a, b, c \in L$ and a, b, c are concurrent$\}$.

Then (L, κ) is an incidence structure called a *projective incidence structure.* A projective incidence structure is called *Pappian* or *Desarguesian* if the theorem of Pappas or Desargues, respectively, is valid in the corresponding projective plane.

If L' is a subset of L and L' has at least two elements then (L', κ') is an incidence structure, where κ' is the restriction of κ to $L' \times L' \times L'$ and (L', κ') is called a *substructure* of the incidence structure. There are four important substructures of the projective incidence structure:

(a) $L \setminus L'$ *contains exactly one line. Then* (L', κ') *is called an affine incidence structure.*

(b) $L \setminus L'$ *consists of a point, thus of the set of lines defining a point in the projective incidence structure* (L, κ). *Then* (L', κ') *is called a star-complement.*

(c) $L \setminus L'$ *consists of exactly two distinct points. Then* (L', κ') *is called a double star-complement.*

(d) $L \setminus L'$ *is an oval i.e. a set of lines containing at least three lines such that no three mutually distinct lines are concurrent. We call* (L', κ') *an oval-complement.*

Typical of the equivalence appearing in the lengthy Chapter 6, Complete S-Group Planes, are:

THEOREM 6.15. *For an S-group plane E the following are equivalent up to isomorphisms*:

(a) *E is a Euclidean plane,*

(b) *E is an affine incidence structure,*

(c) *E is a Pappian affine incidence structure,*

(d) *E is a Euclidean coordinate plane.*

THEOREM 6.28. *For an S-group plane E the following are equivalent up to isomorphisms*:

(a) *E is a Minkowskian plane,*

(b) *E is a double star-complement,*
(c) *E is a Pappian double-star complement,*
(d) *E is a Minkowskian coordinate plane.*

The topic of *complete metric planes* is the last I will mention in any detail. A complete metric plane is a triplet (L, κ, Φ) such that (L, κ) is a Δ-connected incidence structure that contains a quadrilateral and Φ is a map of L into the set of axial collineations of (L, κ) such that $a\Phi$ is a collineation with axis a for all a in L and, for $\sigma_a = a\Phi$ and $S = \operatorname{im}\Phi$, the condition $[S]$ is valid.

$$\sigma_a\sigma_b\sigma_c \in S \Leftrightarrow (a, b, c) \in \kappa. \qquad [S]$$

Complete metric planes and metric planes over three-dimensional metric vector spaces are related by the following:

MAIN THEOREM 6.31. *Up to isomorphisms the complete metric planes are all the metric planes over three-dimensional metric vector spaces* (V, Q) *with* $\dim V^{\perp} \leqslant 1$.

Returning again to the subject of S-groups and S-group planes the main theorem in Chapter 7 provides an embedding of any S-group plane with completely connected points in a complete metric plane and the final chapter, Chapter 8, treats the topic of finite S-groups. The latter includes, without proof, some interesting results of Ott relating the structure of the group G to the geometry of the group plane of G.

Having outlined what I consider to be the highlights of the book I'll now provide a few critical remarks. The first concerns the author's definition of the orthogonal group–defined as the group generated by the reflections on anisotropic lines, (on p. 6). Admittedly, it is noted that this definition is not the customary one and, it is natural considering the spirit of S-groups that permeate throughout the book. However, no mention is made of the Cartan-Dieudonné theorem which, of course, would render the equivalence of the author's definition and the customary one. In particular, there is no mention of the exceptional case of Cartan-Dieudonné which is present for a four-dimensional hyperbolic space over a field with two elements.

In the same equivalence class of criticisms of omissions I'll cite the omission of Dieudonné's two classic works **[D1]** and **[D2]**. This is puzzling, especially when one notices the inclusion of T. Y. Lam's *The algebraic theory of quadratic forms*, and O. T. O'Meara's, *Introduction to quadratic forms*, (both of these books are excellent, to be sure, but much less in the spirit of the book under review than those of Dieudonné).

There are numerous allusions to examples that exist rather than providing the examples or even indicating how they are constructed. This is indeed unfortunate in the initial section on Δ-connectedness; for example on p. 14 it is stated..."*the remaining points may be either* 2- *or* 3-Δ-*connected–either case can occur in specially constructed examples*" yet there is no indication as to how to construct these examples. On p. 24 in the section on incidence structures with reflections it is asserted "there exist incidence structures with reflections in which no meaningful orthogonality can be defined." And again the last sentence in Chapter 4 cites an equivalence which holds in an S-group plane that contains a quadrilateral but which does not hold in an arbitrary S-group.

There are no exercises in the book unless one counts the "proof left to the reader" type. Indeed, a good beginning on a set of exercises might include the above cited examples, and others, with comments and, perhaps, hints.

Some errors are noticeable although certainly not in any great number. In the construction of a Euclidean plane on p. 18 K is assumed to be an arbitrary field yet k is chosen to be an element of K such that $-k$ is not a square. Clearly k cannot be quadratically closed. This same error is repeated on p. 23.

In Theorem 4.2 the existence of a line g is implied by the statement of the theorem; yet the proof of the theorem seems to assume that g exists.

There are relatively few typographical errors, a remarkable feat considering the complexity of some notation and the abundance of subscripts.

This book is a moderately good addition to the literature; its good features outweigh its shortcomings. It should be accessible to patient and persistent beginners and no doubt will be a valuable source for future work on the geometric theory of S-groups.

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Shape theory, by Jerzy Dydak and Jack Segal, Lecture Notes in Math., vol. 688, Springer-Verlag, Berlin-Heidelberg-New York, 1978, vi + 150 pp., $11.80.

Shape theory has come to loom large on the horizon of topology. The literature in the area has grown enormously. More and more research papers assume that the reader is familiar with the results and techniques of shape theory. For the reader who does not have this familiarity, but who wishes to learn, there are difficulties. He may struggle through a paper only to find that the results are superseded by more powerful and completely different techniques. Some results have "standard" errors which may or may not be corrected in the literature. What the newcomer will probably find most irritating is the teeming multitude of approaches to shape theory that he will find. Each approach is derived from a particular viewpoint according to the whim of its originator. Some of these approaches are confused and capable of permanently beclouding the mind as the searcher seeks to find the depth that is not there. Some approaches are so abstract that even experienced mathematicians marvel in wonder at the meaning of it all. To those who are