about what is meant by "optimal". Finally, mention should be made of the excellent collection of exercises, some of them challenging numerical projects involving access to a high speed computer.

There are not many references in the literature where one can learn of real control problems without undue strain on credibility. This book is one of them.

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Basic set theory, by Azriel Levy, Springer-Verlag, Berlin, Heidelberg, New York, 1979, xiv + 391 pp., $24.90.

Perhaps the greatest obstacle in teaching elementary set theory is the mathematical logic needed to formalize the axioms. There is nothing inherently difficult about the basic material–the theory of ordinal and cardinal numbers, and the axiom of choice–which every mathematician is expected to know. And most of the axioms of ZF, the system of Zermelo and Fraenkel used most frequently nowadays, can be stated easily and understandably in English. The exception is the axiom scheme of replacement, which when formalized looks like this

$$\forall x_1 \cdots \forall x_n(\forall x\, \forall y\, \forall z(\varphi(x, y, x_1, \ldots, x_n) \wedge \varphi(x, z, x_1, \ldots, x_n) \rightarrow y = z)$$
$$\rightarrow \forall a\, \exists b\, \forall w(w \in b \leftrightarrow \exists x(x \in a \wedge \varphi(x, w, x_1, \ldots, x_n)))).$$

Here φ stands for a formula of the first-order language of set theory; each such φ yields a new instance of the axiom scheme, so there are infinitely many axioms.

Now the idea behind the replacement scheme is quite simple: any correspondence carries sets to sets. The problem is how the "correspondence" is to be specified. The solution, of course, is that the correspondence must be definable from parameters in a way which could be formalized in first-order logic. Unfortunately, many students do not find this completely clear.

Nor is this the only such problem. To take another example, each instance of the principle of definition by transfinite recursion is usually quite clear, yet the formalization of the principle itself (as a theorem scheme) is often confusing.

There are at least two ways of making things clearer. The first is to abandon formal logic and try to state all the axioms in English. This has the advantage of placing set theory on the same footing as other axiomatic theories in mathematics, such as group theory or linear algebra. This "naive" approach, taken by Halmos in **[1]**, for example, is frequently used in low-level introductions to set theory. It is not much used at higher levels, presumably because it is possible to specify correspondences and sets incorrectly, and clever students will find ways to do it.

Nevertheless the naive approach has value for more sophisticated students, too. There is a serious confusion which arises when the student realizes that in Set Theory 301 the axioms of set theory were presented in first-order logic, whereas in Logic 299 first-order logic was developed within set theory. It deepens when the instructor in Set Theory 302 starts to study ZF in ZF. What is at issue is a confusion between the formal languages studied in set theory and the metalanguage in which the axioms of set theory are stated, caused by excessive formalization of the metalanguage. If the axioms are stated in English, the problem usually disappears.

The more common approach is the introduction of proper classes. The "correspondence" of the replacement scheme really ought to be a function but it is usually too large to be a set, so we agree to call it a proper class. This simplifies the statement of the ZF axioms, but it requires a new axiomatic theory of classes. And the logic is still lurking in the background, this time in the axioms asserting that for each first-order formula φ with parameters, if φ involves only quantification over sets then there is a class consisting of all sets satisfying φ. This class-existence scheme may be replaced by finitely many axioms, but whether it is assumed or derived it plays a crucial role in the theory of classes.

The ideal audience for a set theory course, then, would consist of students thoroughly familiar with logic, and it is for such students that Levy has written his new book, *Basic set theory*. He describes it in the author's preface as "a book on a rather advanced level covering the basic material in an unhurried pace".

What makes it advanced is its presumption of logical (and occasionally set-theoretic!) sophistication. For example, classes are introduced not in the context of a formal class theory, but rather as an eliminable device for shortening and simplifying expressions of ZF. For each formula φ of ZF (with parameters) there is a class term A which has the property that "$a \in A$" is synonymous with "$\varphi(a)$". All the details of this translation are worked out very carefully in the section *before* the one presenting the ZF axioms. Of course only the axiom of extensionality is required at this point, but the reader unfamiliar with set theory will surely find these proceedings mysterious.

This way of presenting classes gives a quasi-intuitionistic flavor to many theorems, since one cannot assert $\exists A\varphi(A)$ without producing a definition for A which will work, nor can one assert $\forall A\varphi(A)$ without giving a proof which will work for every definition. One of the prices to be paid comes when the Axiom of Global Choice must be formulated (pp. 177–180). It cannot be expressed as $\exists F$ (F is a global choice function), for this would mean there is a *definable* choice function. Levy is forced to extend the language by a class constant C and then assert that C is a global choice function.

Another illustration of the level of the book is given by the following proof of the pair-set axiom from the other axioms of ZF, which occurs on p. 20. $\exists x(x = x)$ is a logical axiom, so there is a set. By the separation axiom, which follows from the replacement axiom, the empty set exists. Two applications of the power-set axiom produce the set $\{0, \{0\}\}$, and now, given a and b, we can deduce the existence of $\{a, b\}$ from the replacement axiom applied to $(x = 0 \wedge y = a) \vee (x = \{0\} \wedge y = b)$. On the other hand, once the replacement axiom is reformulated in terms of class functions this argument no longer works, and the pair-set axiom must again be postulated.

The references are one of the delights of the book. Levy has succeeded in tracing most of the basic results back to their sources, and many readers will be surprised, as I was, to find theorems dated twenty years earlier than one would have imagined.

The first half of the book, called *Pure set theory*, treats the theory of ordinal and cardinal numbers and the axiom of choice as far as stationary sets and Silver's theorem on exponentiation with singular cardinals. There is a particularly good discussion of cardinal arithmetic in the absence of the axiom of choice. Coverage is generally quite complete, with topics like ordinal arithmetic and variants of the axiom of choice included.

The second half of the book, called *Applications and advanced topics*, is broken into three parts. One develops point-set topology as far as Polish spaces, in preparation for the study of descriptive set theory. One works out the basics of Boolean algebra in preparation for the study of forcing. The last part treats infinitary combinatorics and large cardinals. Since there is no model theory in the book, this account concentrates on weakly compact and ineffable cardinals. Martin's Axiom and $\diamond$ are introduced and applied but not proved consistent.

It seems unlikely that Levy's book could be used for an elementary set theory course, even at the graduate level, unless the instructor was willing to explain the first chapter at considerable length. In practice, students seem to acquire facility in logic and set theory at about the same time, so there is little chance that an elementary set theory class would know enough logic to handle that chapter without help. A more appropriate use would be for advanced students who already know some logic and set theory, but who want to see a careful treatment of the logical underpinnings of set theory. As Levy points out, the other books written at this level tend to rush through the elementary material in order to reach the frontiers as quickly as possible. Also, especially because of the quality of references, the practicing set theorist will find Levy's book a welcome addition to the literature.

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