SPHERICAL FIBRATIONS

BY J. L. NOAKES

ABSTRACT. In [8], [9] we extend some theorems of I. M. James and J. H. C. Whitehead on the homotopy type of spherical fibrations. Here we sketch our results and methods.

1. Introduction. Let $E_1, E_2$ be $q$-sphere Hurewicz fiberings ($q \geq 1$) over the same connected finite CW-complex $B$. We suppose that the spaces $E_1, E_2$ have the same homotopy type ($E_1 \approx E_2$), that $E_2$ has a cross-section, and that $E_2$ is orientable. Then by [9, Theorem 1] $E_1$ is also orientable. We suppose that $B$ is nilpotent [1], and let $E_j(p) \rightarrow B(p)$ be the localization of $E_j \rightarrow B$ at a prime $p$ ($j = 1, 2$).

THEOREM 1. If dim $B < 2q$ then, for any prime $p$, $E_1(p)$ has a cross-section.

In [3] I. M. James and J. H. C. Whitehead prove a similar result for the case where $B$ is a sphere, but where the fibre of $E_1, E_2$ is not necessarily a sphere. We comment on our proof in §2.

THEOREM 2. If $E_1 \approx B \times S^q$ where $E_1$ has a cross-section then $E_1$ is fibre homotopy trivial.

In [4] James and Whitehead take $B$ to be a sphere and prove a similar result. Our proof in [9] uses a counting argument in the spirit of [3], [7]. Comparing this proof with Theorem 1 we find that if $E_1 \approx B \times S^q$ where dim $B < 2q$ then $E_1$ is fibre homotopy trivial. It was a conjecture along these lines by I. M. James that led me to write [8], [9]. I wish to thank Professor James for telling me his conjecture.

Recall that the fibre suspension [6] $\Sigma E \rightarrow B$ of a map $\pi : E \rightarrow B$ is defined as follows. Let $\Sigma E$ be the quotient of $E \times [-1, 1]$ by the relations $(e, -1) \sim (e', -1)$ and $(e, 1) \sim (e', 1)$ for $\pi(e) = \pi(e')$. Then the projection of $\Sigma E$ takes $[e, t]$ to $\pi(e)$. We assume that $E_2$ has the fibre homotopy type of $\Sigma E$ for some $E$.

REMARKS. (i) If dim $B < q$ then this assumption holds automatically.
(ii) If $E_2$ is a fibre bundle then this assumption is equivalent to the requirement that $E_2$ have a cross-section.

Received by the editors November 16, 1979.


© 1980 American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
THEOREM 3. If either $H^q(B; \mathbb{Z})$ is finite or $\pi_q B = 0$ then there is a homotopy equivalence $f: B \rightarrow B$ such that $E_1$ has the fibre homotopy type of the induced fibration $f^* E_2$.

EXAMPLE. Let $\Phi \subseteq \pi_{r+q} S^q (0 < r < q - 1)$ be maximal with respect to the property that if $\alpha, \beta \in \Phi$ satisfy $\alpha + \beta = 0$ then $\alpha = \beta$. It follows from Theorem 3 that there is a bijection from $\Phi$ onto the set of homotopy types of $q$-sphere fibrations over $S^{r+1}$.

Let $S_j^q$ be the fibre of $E_j (j = 1, 2)$.

THEOREM 4. If the pairs of spaces $(E_1, S_1^q), (E_2, S_2^q)$ have the same homotopy type then there is a homotopy equivalence $f: B \rightarrow B$ such that $E_1$ has the fibre homotopy type of $f^* E_2$.

In [4] James and Whitehead prove a similar result for the case where $B$ is a sphere. In [2] S. Y. Husseini considers the situation of Theorem 4, with the additional hypotheses that $\dim B < q$ and $\pi_1 B = 0$. Simple examples show that [2, Theorem 1.1] is false as stated.

2. Methods. Our proofs of Theorems 3 and 4 turn on an elementary construction which is difficult to describe more briefly than in [9]. However, it may be helpful to say what this construction aims to do.

Let $\pi_j: E_j \rightarrow B (j = 1, 2)$ be the projections, let $s_2$ be a cross-section of $E_2$ and let $\gamma: E_2 \rightarrow E_1$ be a homotopy equivalence. (In the proof of Theorem 4 we take $\gamma$ to be a map of pairs.) A calculation shows that $\pi_1 \gamma s_2$ is a homoto equivalence, and our construction aims to replace $\gamma$ by a homotopy equivalence $\theta'$ so that the diagram

\[
\begin{array}{ccc}
E_2 & \xrightarrow{\theta'} & E_1 \\
\downarrow{\pi_2} & & \downarrow{\pi_1} \\
B & \xrightarrow{\pi_1 \gamma s_2} & B
\end{array}
\]

commutes.

The proof of Theorem 1 is easier to summarise. A counting argument
taken from [7] shows that the Gysin sequences of the $E_j(p)$ split

\[ 0 \rightarrow H^r B_1(p) \xrightarrow{\pi_1^*} H^r E_1(p) \xrightarrow{\psi_1} H^{r-q} B_1(p) \rightarrow 0 \]

\[ 0 \rightarrow H^r B_2(p) \xrightarrow{\pi_2^*} H^r E_2(p) \xrightarrow{\psi_2} H^{r-q} B_2(p) \rightarrow 0 \]

for all $r$. Here $r$ is a homotopy inverse of $\gamma$, and we take coefficients in the ring $\mathbb{Z}(p)$ of integers localized at the prime $p$. Our notation does not distinguish a map from its localization.

We choose $a_0 \in H^q E_2(p)$ so that $\psi_2 a_0$ is the identity of $H^0 B_1(p)$. Let $a_2 = a_0 - \pi_2^* s_2^* a_0$. Then we consider two cases separately.

1. There is no exchange at $p$ when $\psi_1 \tau^* a_2$ is a unit of $H^0 B_1(p)$.
2. There is an exchange at $p$ when $\psi_1 \tau^* a_2$ is not a unit of $H^0 B_1(p)$.

When there is no exchange a computation shows that $\pi_1 \gamma s_2 : B \rightarrow B$ is a homotopy equivalence at the prime $p$. It quickly follows that $E_1(p)$ has a cross-section.

When there is an exchange we form the induced fibrations $\pi_1^* E_1(p), E = \iota^* \gamma^* \pi_1^* E_1(p)$ over $E_1(p), E_2(p) (B^{q-1})_p$. Here $B^{q-1}$ is the $(q-1)$-skeleton of $B$, and $\iota$ is the inclusion of $E_2(p) (B^{q-1})_p$ in $E_2(p)$.

Then $\pi_1^* E_1(p)$ has a cross-section and therefore $E$ has a cross-section.

Next a computation using (2) and dim $B < 2q$ shows that $\pi_1 \gamma \iota$ is almost a homotopy equivalence. So the cross-section of $E$ gives rise to a cross-section of $E_1(p)$.

For details we refer to [8].

Theorems 1, 2, 3 and 4 contain as special cases all the results of [4] except the case $r = q$ of [4, Theorem 1.6]. This exception can be dealt with from our standpoint. There remains the question of what happens when neither $E_1$ nor $E_2$ has a cross-section: for background on this we refer to [5].

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN AUSTRALIA, NEDLANDS, WESTERN AUSTRALIA 6009