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The theory of Lie superalgebras; an introduction, by M. Scheunert, Lecture Notes in Math., vol. 716, Springer-Verlag, Berlin-Heidelberg-New York, vi + 271 pp.

A Lie superalgebra, or ($\mathbf{Z}_2$-) graded Lie algebra, is a vector space $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ with a bilinear multiplication, $\langle\ ,\ \rangle$, satisfying the graded versions of the axioms for Lie algebras: if $X \in \mathfrak{G}_\alpha$, $Y \in \mathfrak{G}_\beta$, and $Z \in \mathfrak{G}_\gamma$ ($\alpha, \beta, \gamma \in \{0, 1\}$), then

(1) $\langle X, Y\rangle = (-1)^{\alpha\beta}[Y, X]$ ("graded antisymmetry");

(2) $(-1)^{\alpha\gamma}\langle X, \langle Y, Z\rangle\rangle + (-1)^{\beta\alpha}\langle Y, \langle Z, X\rangle\rangle + (-1)^{\gamma\beta}\langle Z, \langle X, Y\rangle\rangle = 0$ (the "graded Jacobi identity").

Note that $\mathfrak{G}_0$ is a Lie algebra (in the ordinary sense). In what follows, it will always be tacitly assumed that $\mathfrak{G}$ is finite dimensional and is defined over a field of characteristic 0.

The standard example of an ordinary Lie algebra is $gl(n)$, the space of all $n \times n$ matrices, with $[X, Y] = XY - YX$. (For instance, a representation of a Lie algebra is a homomorphism into $gl(n)$.) There is a corresponding standard example of a Lie superalgebra; it, too, is used to define representations. Let $V = V_0 \oplus V_1$ be a $\mathbf{Z}_2$-graded vector space. We define $pl(V) = pl(V)_0 \oplus pl(V)_1$, where

$$pl(V)_0 = \{V \to V, T(V_j) \subseteq V_j, j = 0, 1\};$$
$$pl(V)_1 = \{S: V \to V: S(V_j) \subseteq V_{1-j}, j = 0, 1\};$$

thus $pl(V)_0$ consists of the linear maps on V taking each distinguished subspace to itself, and $pl(V)_1$ consists of the linear maps on V taking each to the other. The multiplication is given as follows: if X, Y are each in $pl(V)_0$ or $pl(V)_1$, where

$$\langle X, Y\rangle = XY - YX \quad \text{if either } X \text{ or } Y \in pl(V)_0;$$
$$\langle X, Y\rangle = XY + YX \text{ if } X, Y \in pl(V)_1.$$

Thus the multiplication in $pl(V)$ consists of both commutators and anticommutators. It is this fact which explains the sudden interest in Lie superalgebras among physicists; they offer a mathematical framework for combining various symmetry theories. (It seems to be somewhere between unclear and dubious, however, whether the resulting supersymmetry theories do jibe with experimental results. In physics, it is not enough that your theory be elegant; in addition, God must have had the same idea as you.)

Spurred on in large part by the physicists, mathematicians have been working actively in Lie superalgebras. (Actually, Lie superalgebras had arisen some years ago in mathematics–see, e.g., **[4]**–but the subject had languished.) The greatest triumph of this work, and the major topic of Scheunert's book, is V. Kac's classification of the simple finite-dimensional Lie superalgebras over an algebraically closed field of characteristic 0. Kac's result is found in **[1]**; as the book makes clear, **[1]** is by far the most important paper yet on the subject of Lie superalgebras.

The analysis of Lie superalgebras begins with modification of standard Lie algebraic machinery: the Poincaré-Birkhoff-Witt Theorem, the Killing form, and so forth. The subject becomes interesting because it turns out that these tools are only of limited effectiveness. The Killing form, for example, is nondegenerate for all semisimple Lie algebras (and for semisimple Lie algebras only), but its graded analogue can be degenerate for Lie superalgebras.

Scheunert's proof of Kac's theorem follows the outlines of Kac's original proof, but differs in some details. The analysis is in two parts. The graded Jacobi identity implies (among other things) that $\mathfrak{G}_1$ is a $\mathfrak{G}_0$-module (in the usual Lie algebra sense). Call the simple Lie superalgebra $\mathfrak{G}$ *classical* if $\mathfrak{G}_1$ is completely reducible. It turns out that $\mathfrak{G}$ is classical iff $\mathfrak{G}_0$ is reductive. Then, of course, standard Lie algebra representation theory gives some information about $\mathfrak{G}_1$, and a careful analysis (based on the graded Jacobi identity) gives the classification of these algebras. (The procedure given is adapted from **[5]**.)

The nonclassical superalgebras are more difficult. One begins with an analysis of **Z**-graded Lie algebras. $\mathfrak{L} = \bigoplus_{n\in\mathbf{Z}}\mathfrak{L}_{(n)}$ is a **Z**-graded Lie algebra if it has a bilinear multiplication, $\langle\ ,\ \rangle$, such that

(1) $\langle \mathfrak{L}_{(m)}, \mathfrak{L}_{(m)}\rangle \subseteq \mathfrak{L}_{m+n}$;

(2) if $\mathfrak{G}_0 = \bigoplus_{n\in\mathbf{Z}}\mathfrak{L}_{(2n)}$, $\mathfrak{G}_1 = \bigoplus_{n\in\mathbf{Z}}\mathfrak{L}_{(2n+1)}$, then $\mathfrak{L} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ is a Lie superalgebra. One says that $\mathfrak{L}$ is transitive if $\langle \mathfrak{L}_{(-1)}, \mathfrak{L}_{(j)}\rangle \neq \{0\}$ unless $\mathfrak{L}_{(j)} = \{0\}$ and that $\mathfrak{L}$ is irreducible if the obvious representation of $\mathfrak{L}_{(0)}$ in $\mathfrak{L}_{(-1)}$ is irreducible. To classify the nonclassical superalgebras, one first shows that each such superalgebra $\mathfrak{G}$ gives rise to a transitive irreducible **Z**-graded Lie algebra, $\mathfrak{L}$, such that $\mathfrak{L}_{(n)} = 0$ if $n < -1$. One then classifies such **Z**-graded Lie algebras, and uses this classification to determine $\mathfrak{G}$. The classification of the algebras $\mathfrak{L}$ is, like that for classical Lie algebras, a complicated exercise in the representation theory of semisimple Lie algebra, since $\mathfrak{L}_{(0)}$ is reductive. This analysis is taken from Kac **[1]**. Presumably it will be simplified in time.

What else needs to be done? The last chapter of the book gives an introduction to some of the obvious topics. Kac has shown that $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$ is solvable if and only if $\mathfrak{G}_0$ is; he has also shown that the analogue of Lie's theorem (for solvable Lie algebras) is false. There is, presumably, more that can be said about solvable Lie algebras. The representation theory of simple Lie superalgebras promises to be somewhat complicated; for instance, the analogue of H. Weyl's theorem (on the complete reducibility of finite-dimensional representations) is in general false, as is shown in **[1]**. Kac gives a number of results on representations of graded Lie superalgebras in **[1]**; more recently, he has investigated "typical" irreducible representations of simple

Lie algebras, in [2]. Another topic that deserves attention is that of the cohomology of Lie superalgebras. Levi's theorem says that every Lie algebra over C is a semidirect product of its radical with a semisimple Lie algebra. This theorem is generally regarded as a theorem about the vanishing of certain cohomology groups. The analogous theorem for Lie superalgebras is false (as Scheunert remarks); presumably there are interesting results to be found in the cohomology theory of Lie superalgebras. [3] gives a (very) basic introduction to the subject.

The book is clearly written, with very few misprints. It lacks a symbol table, and the index is sketchy; both of these faults are, however, bearable. Scheunert's book should prove a convenient source for information on Lie superalgebras; perhaps it will stimulate further research as well.

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