BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

AN EXAMPLE OF A FIXED POINT FREE HOMEOMORPHISM OF THE PLANE WITH BOUNDED ORBITS

BY STEPHANIE M. BOYLES[1]

In 1912 L. E. J. Brouwer proved his famous translation theorem [3] which states that if h is an orientation preserving homeomorphism of E^2 onto itself having no fixed points, then h is a translation. By a translation, Brower meant that for each x in E^2, $h^n(x) \longrightarrow \infty$ as $n \longrightarrow \pm \infty$; that is, the orbit of every point is unbounded. The question arose as to whether or not any homeomorphism of E^2 onto itself with the property that the orbits of every point is bounded must have a fixed point. This eventually became known as the bounded orbit problem [2].

In this short note we wish to announce the existence of an orientation reversing fixed point free homeomorphism h of E^2 onto itself having the property that the orbit of every point is bounded [1]. We note that the orbit of a point p is the set of all iterates $h^n(p)$, where n is an integer. The homeomorphism we construct can be briefly described as follows. On the complement of the strip $|x| < 1$, h is a reflection across the y-axis. Between the lines $x = -1$ and $x = 1$ we first define h on positive images of the arc $A = \{(x, y)\colon\ |x| \leqslant 1 \text{ and } y = 0\}$. For all integers $m \geqslant 0$ and $k \geqslant 1$, let

$$v_{\pm m,0} = (\pm m/(m + 1), 0),$$

and

$$v_{\pm m,k} = (\pm m/(m + 1), \sum_{i=1}^{k} 1/(m + i)).$$

For all integers j and nonnegative integers k, define

$$h(v_{j,k}) = v_{(-1)^{k+1}-j,k+1}.$$

Extend h linearly on each line segment $[v_{j-1,k}, v_{j,k}]$ by defining

$$h([v_{j-1,k}, v_{j,k}]) = [h(v_{j-1,k}), h(v_{j,k})],$$

Received by the editors May 5, 1980.

1980 *Mathematics Subject Classification*. Primary 54H25, 55M20.

Key words and phrases. Fixed point free homeomorphism, bounded orbits, orientation preserving, orientation reversing.

[1]This is part of the author's doctoral dissertation at the University of Florida under the supervision of Professor Gerhard X. Ritter.

0002-9904/80/00000506/$01.75

for all integers j and nonnegative integers k (Figure 1). Observe that for nonnegative j,

$$h^k(v_{j,0}) = v_{(-1)^k(j+k),k} = ((-1)^k(j+k)/(j+k+1), \sum_{i=1}^{k} 1/(j+k+i)).$$

Since $\Sigma_{i=1}^{k} 1/(j+k+i) \longrightarrow \log 2$ as $k \longrightarrow +\infty$, the images of points of form $v_{|j|,0}$ are bounded. When j is negative,

$$h^{k-j}(v_{j,0}) = v_{(-1)^k k,k-j}$$

forcing all points of form $v_{j,0}$ to have bounded orbits. Hence, with this construction every point on A has a bounded orbit even though the orbit of A itself is unbounded.

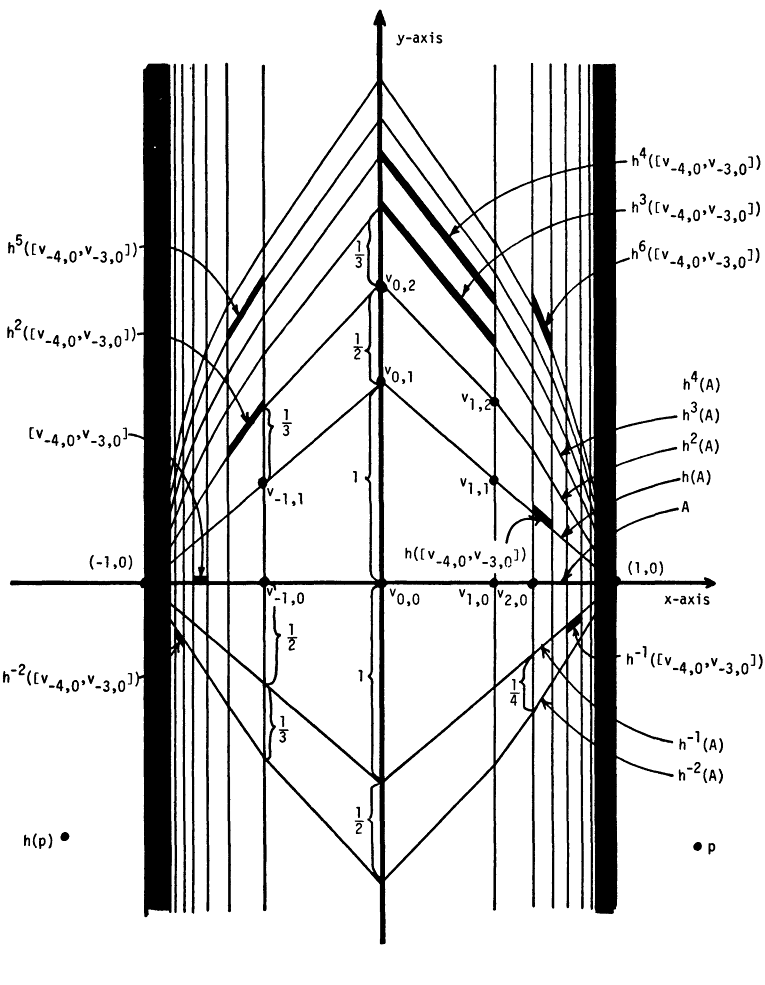

FIGURE 1

To define the homeomorphism between $h^{k-1}(A)$ and $h^k(A)$ so that points on these regions have bounded orbits, we use the fact that the positive orbits of points on $h^k(A)$ are bounded. The homeomorphism is constructed so that under successive applications of h, points between $h^{k-1}(A)$ and $h^k(A)$ are pushed away from the images of $h^{k-1}(A)$ and toward the images of $h^k(A)$. For every point p, there is an integer n (depending on p) such that the distance from $h^n(p)$ to $h^n(h^{k-1}(A))$ exceeds the distance from $h^n(p)$ to $h^n(h^k(A))$. Letting q denote the point on $h^n(h^k(A))$ directly above $h^n(p)$, h is defined so that $h^m(h^n(p))$ lies directly beneath $h^m(q)$ for every nonnegative integer m. Thus, since the positive orbit of q is bounded, the positive orbit of $h^n(p)$ must also be bounded.

For a point p below the x-axis, we define $h(p) = -h^{-1}(-p)$. Hence, points below the x-axis are controlled under applications of h^{-1} in the same way that points above the x-axis are controlled under applications of h.

An open question relating to this problem can be stated as follows. Let h be an orientation reversing homeomorphism of E^2 onto itself having the property that the orbit of every point is bounded. Define h to be *bounded at p* if there exists an open set U containing the point p such that the orbit of U is bounded. Let D be the set of all points p such that h is bounded at p. It can be shown that D is dense and open in E^2. In the example we give, D has infinitely many components. The question as to whether D can have finitely many components without h having a fixed point remains unsolved.[2] Even when D has just two components the answer to this question is not known.

BIBLIOGRAPHY

1. S. Boyles, *A counterexample to the bounded orbit conjecture*, Trans. Amer. Math. Soc. (to appear).

2. B. L. Brechner and R. D. Mauldin, *Homeomorphisms of the plane*, Pacific J. Math. 59 (1975), 375–381. MR 52 #9199.

3. L. E. J. Brouwer, *Beweis des ebenen Translationesatzes*, Math. Ann. 72 (1912), 39–54.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611

[2]This problem was communicated to me by Professor R. Dan Mauldin.