A POINCARÉ-HOPF TYPE THEOREM FOR THE DE RHAM INVARINAT
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The Poincaré-Hopf theorem relates the Euler-characteristic of a manifold to the local behavior of a generic vector-field on the manifold in a neighborhood of its zeroes. As a corollary of this, by taking the gradient, one can calculate the Euler-characteristic of a manifold from a local knowledge of a generic map to \( R^1 \) around its singular points. We prove an analogue of this theorem for calculation of the de Rham invariant of \( 4k + 1 \) dimensional orientable manifolds from a map to \( R^2 \).

For \( 4k + 1 \) dimensional orientable manifolds we have the de Rham invariant \( d(m) \). This invariant is

(a) the rank of the 2-torsion in \( H_{2k}(M) \),

(b) \( \hat{\chi}_2(M) - \hat{\chi}_2(M) \mod 2 \) where \( \hat{\chi}_F(M) \) is the semicharacteristic of \( M \) with coefficients in \( F \),

(c) \( d(M) = [w_2, w_{4k-1}, [M]] = [v_{2k}, q^1 v_{2k}, [M]] \),

where \( w_i(M) \) is the \( i \)th Stiefel-Whitney class and \( v_i \) is the \( i \)th Wu class of \( M \).

For the equivalence of these definitions see [L-M-P]. The de Rham invariant is important in the theory of surgery; see [M] or [M-S].

Definition of the local invariant. Let \( M^m, N^n \) be \( C^\infty \) manifolds. Let \( C^\infty(M, N) \) be the space of \( C^\infty \) maps from \( M \) to \( N \) topologized with the \( C^\infty \) topology. Within \( C^\infty(M, N) \) we have a dense (in fact residual) subset \( G(M, N) \) of maps which are generic in the sense of Thom-Boardman [B] and satisfy the normal crossing condition [G-G]. This second condition is essentially that \( f \) is in general position as a map of its singularity submanifolds to \( N \).

Let \( f \in G(M, R^2) \); then \( df \) is of rank 2 except on a collection of disjoint closed curves in \( M \), the singular set of \( f, S_1(f) \). At points of \( S_1(f) \), \( df \) is of rank 1. Restricted to \( S_1(f) \), \( f \) is an immersion except at a finite set of points, \( S_{1,1}(f) \), the cusp points of \( f \). \( S_1(f) - S_{1,1}(f) = S_{1,0}(f) \) is the set of fold points of \( f \). Suppose \( x \in S_{1,0}(f) \) then we can choose coordinates \( x_1, \ldots, x_n \) around \( x \) and coordinates \( y_1, y_2 \) around \( f(x) \) so that

\[
f(x_1, \ldots, x_n) = (x_1, x_2^2 + x_3^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2).
\]
Similarly if \( x \) is a cusp point we can choose coordinates so that
\[
f(x_1, \ldots, x_n) = (x_1, x_2^3 + x_1x_2 + x_3^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2).
\]

Now we quote a result from [L].

**Theorem.** Let \( M^m, m > 2 \) be of even Euler characteristic; then given \( f \in G(M, R^2) \), \( f \) is homotopic to an \( f_1 \) in \( G(M, R^2) \) with no cusp points.

In the case that \( f \in G(M, R^2) \) has no cusps \( f|S_1(f) \) is an immersion. The normal crossing condition guarantees that \( f(S_1(f)) \) crosses itself in a finite number of double points with no triple points. Let
\[
V(f) = \{ y \in R^2 | f^{-1}(y) \cap S_1(f) = 2 \text{ points} \}.
\]

Let \( N(S_1(f), M) \) be the normal bundle to \( S_1(f) \) in \( M \) and let \( G \) be the bundle over \( S_1(f) \) defined by the following exact sequence:
\[
T(M)|S_1(f) \xrightarrow{df} f^*T(R^2) \rightarrow G \rightarrow 0
\]
where if \( M \) is a manifold \( T(M) \) denotes its tangent bundle. Use of the (second) intrinsic derivative [L], [B] allows definition of a symmetric bilinear form
\[
B: N(S_1(f), M) \otimes N(S_1(f), M) \rightarrow G
\]
on \( N(S_1(f), M) \) with values in \( G \). \( B \) is nondegenerate on \( S_{1,0}(f) \) and has one-dimensional kernel on \( S_{1,1},0 \). Given \( x \in S_{1,0}(f) \) and a choice of an orientation of \( G_x, B_x \) is a nondegenerate bilinear form on \( N(S_1(f), M)_x \). Define the absolute index \( a(x) \) at \( x \) by
\[
a(x) = \min(\text{index}(B_x), m - 1 - \text{index}(B_x)).
\]
Note that \( a(x) \) is independent of the choice of orientation of \( G_x \).

Explicitly suppose \( C_j \) is a component of \( S_1(f) \) with no cusps, then \( f|C_j \) is an immersion so \( N(f(C_j), R^2) \) the normal bundle to \( F(C_j) \) in \( R^2 \) is well defined, isomorphic to \( G \), and trivializable. Let \( T: N(f(C_j), R^2) \rightarrow R^1 \) be a trivialization.

Let \( D(C_j, M) \) be a choice of normal disc bundle to \( C_j \) in \( M \). By a slight abuse of notation we consider \( f: D(C_j, M) \rightarrow N(f(C_j), R^2) \); let \( g_x: D_x(C_j, M) \rightarrow R^1 \) denote the function
\[
g_x = T \circ f: D_x(C_j, M) \rightarrow R^1.
\]
Then \( \{ g_x | x \in C_j \} \) is a differentiable family of functions on the fibers of \( D(C_j, M) \), each \( g_x \) has a Morse singularity at \( x \) and the form \( B_x \) is given by \( d^2g_x \).

Let \( M^m \) be orientable with \( n = 2k + 1 \); then \( M \) has zero Euler-characteristic so use the theorem of Levine to choose \( f \in G(M, R^2) \) with no cusps. Let \( C_j \) be a component of \( S_1(f) \); as \( M \) is orientable \( N(C_j, M) \) is trivializable. Furthermore the choice of trivialization \( T \) above makes \( B \) a nondegenerate symmetric bilinear form on \( N(C_j, M) \). Thus the structure group of \( N(C_j, M) \) is given a reduction to \( O^+(p, m - p - 1) \), the orientation preserving components of
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O(p, m — p — 1). For p ≠ 0, m — 1, O+(p, m — p — 1) has two components. Define
i(Cj) ∈ Z/2Z, the index of Cj, by i(Cj) = 0 if and only if N(Cj, M) is the trivial
O+(p, m — p — 1) bundle and i(Cj) = 1 if and only if N(Cj, M) is the nontrivial
O+(p, m — p — 1) bundle. Note that i(Cj) is independent of all choices of triviali-
zation and orientations.

Define τ(f) ∈ Z/2Z by

τ(f) = ∑ i(Cj), Cj a component of S1(f).

Define r(f) = |V(f)| mod 2.

Statement of results.

PROPOSITION A. r(f) = τ(f) + τ(f) is independent of the choice of f ∈ G(M, R2) without cusps, so one can define r(M) = r(f), f ∈ G(M, R2) without cusps.

COMMENT. One can, with a little more effort, still define r(f) for f ∈ G(M, R2) even when f ∈ G(M, R2) has cusps. However in this case r(f) is no longer independent of f.

PROPOSITION B. r is a homomorphism from oriented cobordism to Z/2Z; that is

(a) If [M] = [N] in Ω2k+1 then r(M) = r(N),
(b) r(M1 ∪ M2) = r(M1) + r(M2).

Let χ(M) be the Euler characteristic of M reduced mod 2.
(c) r(M2k+1 × N2p) = r(M2k+1) · χ(N2p).

PROPOSITION C.

(a) Let M4k+1 be an orientable manifold then r(M) = d(M).
(b) Let M4k+3 be an orientable manifold then r(M) = 0.

Thus Proposition C gives a way of determining the de Rham invariant, which is intersection theoretic in character, from the local behavior of a map M to R2 around its singular set. It is illuminating to consider such an f as a pair of Morse functions in general position with respect to each other.

Sketch of proofs. Proposition A is proved by a careful analysis of a homotopy F from f0 to f1 where f0 and f1 are different choices of f on M2k+1. We can take F ∈ G(M × I, R2 × I). First we reduce to the case that F has no dovetail singularities. In this case S1(F) is an embedded surface in M × I intersecting M × {0} and M × {1} normally in S1(f0) and S1(f1). On the interior of this surface we have circles of cusp points separating the surface into regions of constant absolute index. Let Rp be the union of the regions of absolute index p. Let i(Rp) = Σ i(C) C a component of ∂(Rp), then analysis of the cusp
singularity yields the equation
\[ \sum_{i=0}^{k} i(R_p) = \tau(f_0) + \tau(f_1). \]

For \( p \neq k \) it is straightforward to prove \( i(R_p) = 0 \). As in the case that \( F \) has no dovetails we have
\[ t(f_0) = t(f_1) \mod 2. \]

Proposition A reduces to showing \( i(R_k) = 0 \). This is easy for \( k \) odd, but subtler for \( k \) even. For \( k \) even we prove

**LEMMA.** Let \( P \) be a component of \( R_k \); then \( P \) is a closed surface \( P^1 \) minus a collection of discs and \( P^1 \) is of even Euler characteristic. From this fact it follows from \( i(R_k) = 0 \).

Proposition B is proved by first observing that \( r(M) \) remains invariant if \( M \) is cut open and repasted along a codimension 1 submanifold of the form \( S^1 \times F \) by a pasting \( \phi: S^1 \times F \to S^1 \times F \) with \( \phi(x \times F) = x \times F \). Given this observation the results of [A] allow immediate demonstration of bordism invariance. For the relation of cutting and pasting and cobordism see [K-K-N-0].

Proposition C follows from explicit construction of the examples (using, for instance, (c) of Proposition B) in each dimension \( 4k + 1 \) on which \( r \) and \( d \) agree and are nonzero. This, in addition to the result of [Br] that \( d(M) \) vanishes if and only if \( [M] \) has a representative fibered over the two-sphere, is enough to show \( r(M) \) and \( d \) have the same kernel and same range and hence agree. Finally to show \( r(M^{4k+3}) = 0 \) we use the results of [A-K] to choose a representative of \( [M] \) fibered over \( S^2 \). On such a representative \( r(M) \) is zero by the observation of the previous paragraph.

REFERENCES


[A-K] J. C. Alexander and S. M. Kahn, Characteristic number obstructions to fibering oriented and complex manifolds over surfaces, Univ. of Maryland, College Park, Maryland (Preprint).


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