BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 4, Number 1, 1981

SMOOTH BOUNDED STRICTLY AND WEAKLY PSEUDOCONVEX DOMAINS CANNOT BE BIHOLOMORPHIC

BY STEVEN BELL

There is no Riemann mapping theorem in the theory of functions of several complex variables; nor is there a Riemann nonmapping theorem. In fact, until recently, it was not known whether it is possible for a smooth bounded strictly pseudoconvex domain to be biholomorphically equivalent to a smooth bounded weakly pseudoconvex domain. The paper [1] answers this question in the negative by proving

THEOREM 1. *If D_2 is a smooth bounded pseudoconvex domain, and D_1 is a smooth bounded domain whose $\bar{\partial}$-Neumann problem satisfies global regularity estimates, then biholomorphic mappings between D_1 and D_2 extend smoothly to the boundary.*

In particular, it is known (Kohn [6]) that the $\bar{\partial}$-Neumann problem satisfies global regularity estimates in smooth bounded strictly pseudoconvex domains. Hence, a biholomorphic mapping between a smooth bounded strictly pseudoconvex domain and a smooth bounded weakly pseudoconvex domain would extend smoothly to the boundary. Since strict pseudoconvexity is preserved under biholomorphic mappings which extend to be C^2 up to the boundary, both domains must be strictly pseudoconvex.

Other domains for which the $\bar{\partial}$-Neumann problem is known to satisfy global regularity estimates include smooth bounded weakly pseudoconvex domains with real analytic boundaries [7], [2] and certain domains of finite type [7].

The proof of Theorem 1 exploits the transformation rule for the Bergman projection. If P_i denotes the Bergman orthogonal projection of $L^2(D_i)$ onto its subspace of holomorphic functions, $i = 1, 2$, and if $f\colon D_1 \longrightarrow D_2$ is a biholomorphic mapping, then

$$P_1(u \cdot (\phi \circ f)) = u \cdot ((P_2\phi) \circ f)$$

where $u = \mathrm{Det}[f']$ and $\phi \in L^2(D_2)$. It is possible to construct functions ϕ which vanish to arbitrarily high order on bD_2 such that $P_2\phi \equiv 1$. If ϕ is such a function which vanishes to a high enough order on bD_2 to make $u \cdot (\phi \circ f)$ smooth up to the boundary, then u is the projection of a function which is smooth up to the

Received by the editors August 1, 1980.
1980 *Mathematics Subject Classification.* Primary 32H99; Secondary 35N15, 32A40.

boundary. Since global estimates for the $\bar{\partial}$-problem in D_1 imply global estimates for P_1, we conclude that u is smooth up to the boundary.

A similar argument reveals that $u \cdot (g \circ f)$ is smooth up to the boundary whenever g is a holomorphic function on D_2 which is smooth up to the boundary.

The final steps in the proof of Theorem 1 depend on Kohn's theory of the $\bar{\partial}$-Neumann problem with weight functions [8] and a special Sobolev inequality for holomorphic functions: If g and h are holomorphic functions on a smooth bounded domain D, then

$$\left| \int_D h\bar{g} \right| \leqslant C \|h\|_s \, \|g\|_{-s}$$

where $\|h\|_s^2 = \Sigma_{|\alpha| \leqslant s} \|D^\alpha h\|^2_{L^2(D)}$ is the usual Sobolev s-norm and

$$\|g\|_{-s} = \sup_{\substack{\phi \in C_0^\infty(D) \\ \|\phi\|_s = 1}} |\langle g, \phi \rangle_{L^2(D)}|.$$

The constant C only depends on D and the positive integer s. This inequality is unique to holomorphic functions and must not be confused with the standard generalized Schwarz inequality.

Theorem 1 generalizes and improves several earlier results on biholomorphic mappings due to Henkin [5], Pinčuk [9], Fefferman [4], Diederich and Fornaess [3]. The Henkin, Pinčuk, Diederich and Fornaess results also apply to proper holomorphic mappings. At the present time, the techniques used in the proof of Theorem 1 do not seem to adapt to the proper mapping case.

REFERENCES

1. S. Bell, *Biholomorphic mappings and the $\bar{\partial}$-problem*, Ann. of Math. (2) (to appear).

2. K. Diederich and J. E. Fornaess, *Pseudoconvex domains with real-analytic boundary*, Ann. of Math. (2) **107** (1978), 371–384.

3. ———, *Proper holomorphic maps onto pseudoconvex domains with real analytic boundary*, Ann. of Math. (2) **110** (1979), 575–592.

4. C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. 26 (1974), 1–65.

5. G. Henkin, *An analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain*, Soviet Math. Dokl. **14** (1973), 858–862.

6. J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds*. I, II, Ann of Math. (2) **78** (1963), 112–148; **79** (1964), 450–472.

7. ———, *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions*, Acta Math. **142** (1979), 79–122.

8. ———, *Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc. **181** (1973), 273–292.

9. S. Pinčuk, *On proper holomorphic mappings of strictly pseudoconvex domains*, Siberian Math. J. **15** (1974), 644–649.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540