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FOR $n > 3$ THERE IS ONLY ONE FINITELY ADDITIVE ROTATIONALLY INVARIANT MEASURE ON THE n-SPHERE DEFINED ON ALL LEBESGUE MEASURABLE SUBSETS

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The following paragraph is taken from the introduction of Joseph Rosenblatt's paper **[R]**.

"Let β be the ring of Lebesgue measurable sets in the n-sphere S^n, and let λ_n denote the Lebesgue measure on β normalized by $\lambda_n(S^n) = 1$. The classical characterization by Lebesgue of λ_n is that it is the unique positive real-valued function μ on β which satisfies these three conditions:

(a) $\mu(S^n) = 1$;

(b) μ is invariant under isometries;

(c) μ is countably additive.

In 1923 Banach **[B]** studied the question of Ruziewicz whether μ is still unique when (c) is replaced by

(c_0) μ is finitely additive.

Banach gave a negative answer to this question for S^1 but for S^n, $n \geqslant 2$, the question is still unanswered."

From the body of Rosenblatt's paper one can extract the implication that *if Lebesgue measure λ_n on S^n is not characterized by* (a), (b), *and* (c_0) *then there is a net of measurable subsets $(A_\alpha) \subset S^2$ which is asymptotically invariant and nontrivial, namely* $\lim_\alpha(\lambda_n(gA_\alpha \Delta A_\alpha)/\lambda_n A_\alpha) = 0$ *for all rotations g and so that* $0 < \lambda_n(A_\alpha) \leqslant c < 1$ (Theorem 1.4 of **[R]**). Here $A \Delta B = A \cup B - A \cap B$.

The following Proposition will show that such asymptotically invariant nets on S^n are impossible, $n > 3$.

PROPOSITION. *For each $n > 3$ there is a countable subgroup Γ_n in the group O_{n+1} of rotations of S^n satisfying*

(i) *the action of Γ_n on S^n is ergodic,*

(ii) *the group Γ_n satisfies Kazhdan's property T:*

There exist a finite subset $\Lambda \subset \Gamma_n$ and an $\epsilon > 0$, so that for any unitary representation π if Γ, if there exists a vector ζ in H_π such that $\|\zeta\| = 1$,

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$\|\pi(g)\zeta - \zeta\| \leqslant \epsilon$ $\forall g$ in Λ then there exists a vector $\zeta' \in H_\pi$ with $\pi(g)\zeta' = \zeta'$ $\forall g \in \Gamma_n$, and $\zeta' \neq 0$.

PROOF. For $n > 3$ let Γ_n be the group of $(n + 1) \times (n + 1)$ matrices with entries integers $(n + m\sqrt{2})$ of the field $Q(\sqrt{2})$ where such matrices preserve the quadratic form

$$x_0^2 + x_1^2 + \cdots + x_{n-2}^2 - \sqrt{2}x_{n-1}^2 - \sqrt{2}x_n^2.$$

If we conjugate all the matrices of Γ_n by the field automorphism of $Q(\sqrt{2})$ we obtain a group of matrices isomorphic to Γ_n preserving the form

$$x_0^2 + x_1^2 + \cdots + x_{n-2}^2 + \sqrt{2}x_{n-1}^2 + \sqrt{2}x_n^2.$$

So Γ_n is embedded as a subgroup of $O(n + 1)$, the real orthogonal group of the second quadratic form. If $O(n - 1, 2)$ denotes the real orthogonal group of the first quadratic form then the diagonal embedding $\Gamma_n \longrightarrow O(n + 1) \times O(n - 1, 2)$ is discrete because the diagonal embedding $(n + m\sqrt{2}) \in Q(\sqrt{2}) \longrightarrow (n + m\sqrt{2}, n - m\sqrt{2}) \in R \times R$ is discrete. By a basic theorem of arithmetic groups Γ_n has cofinite volume in $O(n + 1) \times O(n - 1, 2)$. Since $O(n + 1)$ is compact, Γ_n is discrete with cofinite volume in $O(n - 1, 2)$.

Since $O(n - 1, 2)$ is a simple Lie group of real rank $\geqslant 2$ it has Kazhdan's property (see **[K]**) which descends by an averaging argument (Theorem 3 of **[K]**) to the discrete subgroup with cofinite volume Γ_n. Thus Γ_n has Kazhdan's property T. This proves (ii).

Now if the topological closure of $\Gamma_n \subset O(n + 1)$ were a proper closed subgroup G then the complexification $G_{\mathbf{C}}$ of G in the complexification $O(n + 1, \mathbf{C})$ of $O(n + 1)$ would define a proper $\mathbf{C}$-algebraic subgroup containing Γ_n. But for the conjugate embedding $\Gamma_n \subset O(n - 1, 2) \subset O(n + 1, \mathbf{C})$, Γ_n is Zariski dense by Borel's density theorem. This is a contradiction showing Γ_n is topologically dense in $O(n + 1)$.

Since Γ_n is a dense subgroup of isometries ergodicity follows immediately from a consideration of Lebesgue density points. This proves (i).

Combining the Proposition with Rosenblatt's work **[R]** we have the answer to the Banach-Ruziewicz problem, $n > 3$.

THEOREM. *Spherical measure on S^n, $n > 3$, is the only finitely additive normalized measure invariant under rotations and defined*[1] *on all Lebesgue measurable sets.*

PROOF. If not by Rosenblatt **[R]** there is, as mentioned above, a nontrivial asymptotically invariant net of sets $(A_\alpha) \subset S^2$. Clearly we can extract a

[1]In **[R]** one finds Tarski's observation using paradoxical decompositions that if a finitely additive measure is defined on all Lebesgue measurable sets it must be zero on Lebesgue null sets.

countable subsequence $(A_j) \subset S^2$ which is asymptotically invariant for the countable subgroup $\Gamma_n \subset O_{n+1}$ constructed in the Proposition. Namely, $0 < \lambda_n(A_j) \leqslant c < 1$, and for all $g \in \Gamma_n$ $\lim_j(\lambda_n(gA_j \Delta A_j)/\lambda_n(A_j)) = 0$.

Now convert the characteristic $f^{\underline{ns}}$ of A_j into functions of integral zero by forming $f_j = (\chi A_j/\sqrt{\lambda_n(A_j)} - \sqrt{\lambda_n(A_j)})$ and then $F_j = f_j/\|f_j\|_2$. Then $\int F_j d\lambda_n = 0$ because $\int f_j d\lambda_n = 0$, and $\|F_j\|_2 = 1$. Also

$$\|f_j \circ g - f_j\|_2^2 = 2(1 - \lambda_n(g^{-1}A_j \cap A_j)/\lambda_n(A_j)).$$

Since $\|f_\alpha\|_2^2 = \lambda_n(S^2 \mid A_j)$, it is bounded away from zero by the nontriviality of (A_j). Thus $\lim_j\|F_j \circ g - F_j\|_2 = 0$ for all $g \in \Gamma_n$ and $\|F_j\|_2 = 1$. (Compare [**R**, Lemma 3.1].)

If we apply property T for Γ_n for the representation of Γ_n on the space H of square integrable $f^{\underline{ns}}$ on S^n of integral zero we obtain from the existence of the vectors F_j of H, the existence of an element in H of norm 1 which is Γ_n invariant. This contradicts ergodicity of Γ_n. Thus there is no such net of asymptotically invariant sets, and the Theorem is proved.

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REFERENCES

[B] S. Banach, *Sur le problème de la mesure,* S. Banach Oeuvres, vol I, Warszawa, 1967, pp. 318–322.

[K] D. A. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups,* Functional Anal. Appl. 1 (1967), 63–65.

[R] J. Rosenblatt, *Uniqueness of invariant means for measure preserving transformations,* Trans. Amer. Math. Soc. (to appear).

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