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BOOK REVIEWS

Projective geometries over finite fields, by J. W. P. Hirschfeld, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979, xii + 474 pp., $45.00.

It is a commonly held misconception that little research in the area of finite Desarguesian geometry is going on at present. While it is true that classical geometry as a discipline has been far less in vogue than it was at the beginning of this century, and that a great deal of modern research concentrates on non-Desarguesian planes, the classical case has not been ignored by all. Finite geometry in particular, inspired by the resurgence of combinatorial theory, very active work in finite algebra, and its own intrinsic appeal, has been given much attention.

Classically, $PG(n, K)$, the projective space of finite dimension n over a field K, is developed from a set of postulates, as for example in the well-known and still useful 1910 volume by Veblen and Young (*Projective geometry*, Blaisdell, New York). Nowadays, with linear algebra at hand and well developed, the concept can be introduced very quickly: Let V be the vector space of dimension $n + 1$ over K. Then $PG(n, K)$ is an incidence structure consisting of subspaces of dimension m $(0 \leqslant m < n)$, which are simply $(m + 1)$-dimensional subspaces of V; incidence is defined as inclusion. Subspaces of dimension 0, 1 and 2 are called points, lines and planes respectively. A subspace of maximum dimension $n - 1$ is called a hyperplane. Characteristic properties of projective geometry are easily established; for example, when $n = 2$ any two distinct lines in the projective plane $PG(2, K)$ intersect because any two distinct two-dimensional subspaces in a three-dimensional vector space share a one-dimensional subspace. One proceeds quickly to a study of collineations in $PG(n, K)$, these being nonsingular semilinear transformations in V. The introduction of sesquilinear forms into V, when applied to $PG(n, K)$, reproduces the classical concept of correlations (point-hyperplane correspondences), leading naturally to polarities and a study of quadric surfaces.

Projective geometry received a great deal of attention in the nineteenth and early twentieth centuries, with the concentration being almost exclusively on the geometry over the real and complex fields. The problems considered were mainly such as could be handled by methods of real and complex analysis, although they were in fact often more elegantly treated by synthetic means. Thus in addition to basic properties such as the theorems of Desargues and Pappus, with their consequences, attention was centred on conics and quadric surfaces, and generalizations thereof to curves and surfaces of higher order. In real projective geometry, theorems on order and continuity were of course very important.

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Although traces of geometry over finite fields can be found in the nineteenth century, a systematic study did not come until the present century was well launched. The properties of $PG(n, q)$ ($PG(n, K)$, where $K = GF(q)$, q being a prime power) are of course determined by the properties of $GF(q)$, and therefore the focus of interest is different than it is in real or complex geometry. We still have the theorems of Desargues and Pappus, since these are geometric realizations of properties common to all fields. We still study conics, quadrics, and higher order curves and surfaces, but without the aid of the tools derived from order and limiting processes. Instead we have the much more primitive process of counting, and this radically changes the nature of the study. Arising out of this different perspective, we find the more general concepts of ovals and k-arcs taking on significance. Configurations, while certainly not missing from older projective geometry, come into full flower in the finite case. Thus the study of spreads–sets of $1 + q^{t+1}$ t-dimensional subspaces which partition the points of $PG(2t + 1, q)$–radically generalizes the classical notion of elliptic linear congruences. Projective subspaces of the same dimension but over subfields of $GF(q)$ are important configurations, as are blocking sets, the latter being only definable in finite spaces, and therefore unknown in earlier research. These concepts, developed through combinatorial arguments and the application of the algebra of finite fields, frequently appear in other guises in the Design of Experiments, Coding Theory, and other combinatorics-related subjects, including, pleasantly enough, much more general (non-Desarguesian) geometry. They have provided new insights, and have in turn been further developed themselves by reference to the research carried on in these related areas. There is every reason to believe that this interaction will continue to be fruitful. It is therefore both important and timely that the present volume under review has appeared, to make this research more readily available.

A glance at the twenty-nine pages of bibliography shows a wide variety of contributions in the areas mentioned above, most being dated in the 1960's and 1970's. The author, in his preface, describes his book as "an attempt to collect the results of these papers, and to present a self-contained and comprehensive account of the subject, assuming no more knowledge than the group theory and linear algebra taught in a first degree course, as well as a little projective geometry, and a very little algebraic geometry". In this reviewer's opinion, the attempt was eminently successful.

Unlike the "traditional" book on geometry, there is here no axiomatic approach to the subject. Such an approach is unnecessary, as we have pointed out, since $PG(n, K)$ is easily defined in terms of the vector space of dimension $n + 1$ over K, and this is the approach which Hirschfeld uses. After an introductory chapter summarizing the relevant properties of finite fields, he defines $PG(n, K)$ and proceeds to develop elementary properties of $PG(n, q)$. Such concepts as partitions by subspaces and subgeometries, cyclic projectivities, polarities, and quadric and Hermitian varieties are covered here. Next comes a short analysis of $PG(1, q)$, after which the remainder of the volume–a full two-thirds of it–is devoted to $PG(2, q)$. It is here that properties of k-arcs and of ovals in general are treated in a valuable summary of this field

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of current research. The reviewer was particularly happy to see the inclusion of the next topic: plane cubic curves. This is dealt with in a long chapter, culminating in a classification of the singular and nonsingular cubics in $PG(2, q)$. There follow interesting chapters on plane $(k\colon n)$-arcs and blocking sets, with a final chapter of detailed analysis of $PG(2, q)$ for small values of q.

One very helpful feature of the format of the text is a concluding section of each chapter, entitled "notes and references". Here the credits for the work of each preceding section are given, thus freeing the main text of a plethora of individual references, which would hamper the smooth flow of the development. The book is essentially self-contained and serves well the dual purpose of a good reference source for the expert and a vehicle of self-study for the newcomer to a fascinating but relatively unknown subject.

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Translation planes, by Heinz Lüneburg, Springer-Verlag, Berlin and New York, 1980, ix + 278 pp., $29.80.

As the author says in his foreword, infinite translation planes are not completely disregarded but finite ones form the main theme of the book. Finite projective and affine planes have interest for mathematicians in many other areas. First, they are important for combinatorialists, including people in design of experiments and in coding theory. Many combinatorial structures have natural representations associated with finite projective planes. Projective planes (finite or infinite) have obvious significance for the foundations of geometry. If one were to look for the field of interest whose adherents were most likely to have a strong side interest in projective planes a strong candidate would be the field of finite groups. Many of the early developments in projective planes came from people working on nonassociative division rings (semifields).

The known finite projective (or affine) planes are closely related to translation planes.

One way of looking at the Euclidean plane is to identify the points with elements (x, y) of a two-dimensional vector space over the reals. The lines through the origin are one-dimensional vector spaces; the other lines are translates of the lines through the origin. This book develops the following analogous way of representing a translation plane: Let V be a vector space which is a direct sum of two copies of a subspace V_1 so that $V = V_1 \oplus V_1$. Each element of V can then be represented by an ordered pair (X, Y), $X, Y \in V_1$. If V_1 has dimension r then V has dimension $2r$ and the sets of vectors for which $X = 0$ or for which $Y = 0$ are r-dimensional subspaces isomorphic to V_1. A *spread* defined on V is a class of r-dimensional subspaces (the *components* of the spread) such that each nonzero element belongs to exactly one component and V is the direct sum of each pair of distinct

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