In sharp contrast to Edwards' verbose style, Ribenboim's presentation is clear, concise and elegant. The thirteen lectures cover the major events of all three eras with a heavy emphasis on the post-Kummer era. The book is a true work of art. The lectures are well organized and present the mathematics underlying seemingly isolated results in a very cohesive manner. In order to avoid too much technical detail, the proofs of the more difficult theorems are sometimes only sketched and other times omitted completely. An extensive bibliography is given at the end of each section so the reader can easily locate sources which cover material missing in the text. Ribenboim also promises a second volume which is intended to contain much of the technical development which was omitted in these thirteen lectures. His first book should stimulate interest in and promote a better understanding of the mathematics related to Fermat's last theorem. One can only have high expectations for Ribenboim's second book on this subject.

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Classification theory and the number of non-isomorphic models, by Saharon Shelah, Studies in Logic and the Foundations of Mathematics, Volume 92, North-Holland Publishing Company, Amsterdam-New York, 1978, xvi + 544 pp., $62.25.

Can the dimension theory of vector spaces, algebraically closed fields, countable torsion Abelian groups (Ulm's Theorem) etc. be generalized to provide a means of characterizing the models of an arbitrary first order theory? If not, can the obstacle to such an extension be identified and the program carried through in its absence? A vector space or an algebraically closed field is determined by a single cardinal (the number of independent elements); a countable torsion Abelian group is determined by an infinite sequence of cardinals. Thus by a generalized dimension theory we mean a method of attaching to each model a sequence of cardinals which determine it up to isomorphism. The first test of such a generalized dimension theory is its ability to solve the spectrum problem: i.e., to count the number of models of a theory. In fact, Shelah's answer to these questions arose from the study of the following problem. For a first order theory T, let $n(T, \lambda)$ denote the number of non-isomorphic models of T with cardinality λ. Determine the possible spectrum functions $n(T, \lambda)$. Shelah's book contains the most complete available account of the methods for attacking these problems. It shows that the non-superstability of a theory T is the obstacle to developing dimension for models of that theory. On the one hand, a theory which is not superstable has 2^κ models of cardinality κ (for sufficiently large κ). On the other, he develops a structure theory for models of superstable theories.

There are two basic model theoretic concepts essential to even a vague understanding of the methods of this book: the notion of a formal theory and the idea of a type.

First, for example, consider the formal theory of modules. Fix a ring R. A natural language, L, for formalizing the study of modules over R contains a constant symbol 0, a function symbol $+$, for each element of the ring R a unary operation symbol f_r, and variables ranging over elements of a module. The first order sentences of this language are built inductively. The basic formulas are equations between polynomials. More complicated expressions are built by closing under Boolean operations and under existential and universal quantification over elements of the module. A "first order theory of modules" is a collection of sentences from L. The module M satisfies (or is a model of) the theory T if for each $\phi \in T$, ϕ is true in M (written $M \models \phi$). If $\mathrm{Th}(M)$ denotes the collection of sentences from this language which are true in M, then $\mathrm{Th}(M)$ is complete in the sense that for every sentence ϕ, either ϕ or $\sim \phi$ is in $\mathrm{Th}(M)$. If the ring R is uncountable we have a natural example of a mathematical theory which is formalized in an uncountable language.

To orient the reader we give some examples of first order theories of modules. For simplicity, we take $R = Z$ and describe some theories of Abelian groups. Each of the following classes is the collection of models of a first order theory: (i) torsion free Abelian groups, (ii) torsion free divisible Abelian groups, (iii) Abelian groups of exponent 3. The first theory is not complete; the other two are. In contrast, the collection of torsion Abelian groups is not the class of models of a first order theory of modules.

A type (or element type) describes the relation between a point and a set. In algebra, one can describe the relation between a point and a set by describing the subalgebra generated by the two. In model theory, because there are relations in the language and because the models of an arbitrary theory are not closed under substructure, the relationship is more complicated. A type over a set A contained in a model of a theory T is a complete description of the relation of an element b (generally not in A) to A. Formally, $t(b; A) = \{\phi(x, a) \colon \models \phi(b, a)\}$. The collection of all such types is the Stone space of A, denoted $S(A)$.

The book is more a series of research articles than a text. Much of the material has never before appeared in print and all results are proved in the greatest possible generality. Much of the material is difficult even for an experienced model theorist. In this review we will try to place the book in the context of earlier work on the spectrum problem and sketch a few of the major ideas while skirting all the technical difficulties.

The book considers theories in arbitrary first order languages. For simplicity of notation in this review T has a countable language unless a remark is formulated in terms of $|T|$ (the number of symbols in the language of T). All theories considered will be complete.

To place the book in context, we will now review briefly the pre-Shelah results on the spectrum problem. Since we are dealing with complete theories with an infinite model the problem of determining in which finite cardinals the theory has a model (often called the spectrum problem) does not arise at all. By the Lowenheim-Skolem-Tarski theorem if a theory has an infinite model, it has one of every infinite cardinality. The theory T is categorical in power λ if all models of T with cardinality λ are isomorphic. $\aleph_0$-categorical theories were nicely characterized in the 1950's [6]. Vaught proved in [7] that no countable complete theory has exactly two countable models and reported Ehrenfeucht's examples of theories with exactly n models for each finite $n > 2$. It is easy to construct theories with exactly $\aleph_0$ and exactly continuum many countable models. Vaught conjectured that there are no other possibilities for $n(T, \aleph_0)$. This problem remains open although a number of special cases have been solved, many of them using the methods of this book. In [3] Łos conjectured that in analogy with the theory of algebraically closed fields of fixed characteristic or with the theory of torsion-free divisible abelian groups, a theory is categorical in one uncountable power just if it is categorical in all uncountable powers. M. Morley [4] proved the Łos conjecture and himself conjectured that for every countable T, $n(T, \lambda)$ is a non-decreasing function of λ with one exception: those T, like the theory of algebraically closed fields of fixed characteristic, where $n(T, \aleph_0) = \aleph_0$ and $n(T, \lambda) = 1$ for uncountable λ. This exceptional case was completely investigated by Morley [4], [5] who showed that if $n(T, \kappa) = 1$ for some uncountable κ then $n(T, \kappa) = 1$ for all uncountable κ and $n(T, \aleph_0) \leqslant \aleph_0$ and Baldwin and Lachlan [1] who showed $n(T, \aleph_1) = 1$ implies $n(T, \aleph_0)$ is 1 or $\aleph_0$.

This book makes considerable progress on Morley's conjecture (and its generalization to uncountable languages). The first step is to divide theories into those which have the maximal number of models (i.e. 2^κ if $\kappa \geqslant |T|$) and those which don't. Although a characterization of theories along this line is not complete in this book, the stability hierarchy provides a good approximation thereto. T is stable in λ if for every model, M, of T $|M| \leqslant \lambda$ implies $|S(M)| \leqslant \lambda$. T is stable if T is stable in some λ. Now Shelah proves that for any countable T one of the following holds.

(i) T is stable in all λ (ω-stable).
(ii) T is stable in all $\lambda \geqslant \mathrm{Exp}(2, \aleph_0)$ (superstable).
(iii) T is stable in λ if $\mathrm{Exp}(\lambda, \aleph_0) = \lambda$ (stable).
(iv) T is stable in no λ (unstable).

Each of these classes is absolute in the technical set-theoretic sense. For example, T is unstable just if there is a formula in T which linearly orders an infinite set of n-tuples from some model of T. Figure 1 and the accompanying theorem illustrate the relation between these concepts and the spectrum problem. Such terms as "multidimensional" will be defined below.

For simplicity in describing the following results let $\aleph_\beta \geqslant c$ (the power of the continuum).

THEOREM. *Let T be a countable complete first order theory.*
(i) *If T is not superstable then $n(T, \kappa) = 2^\kappa$ for all uncountable κ.*
(ii) *If T is superstable and multidimensional, $n(T, \aleph_\beta) \geqslant 2^\beta$.*
(iii) *If T is superstable and not multidimensional $n(T, \aleph_\beta) \leqslant (\beta + 1)^c$.*
(iv) *If T is superstable and unidimensional $n(T, \kappa) \leqslant 2^c$ for all κ.*

unstable	Any theory of Linear order arithmetic		
stable but not superstable	$\mathrm{Th}(Z^\omega; +)$		
superstable not ω-stable	a $\mathrm{Th}(Z; +)$	$\mathrm{Th}(Z \otimes Z_3^\omega; +)$	disjoint union of a and b
ω stable	ACF_0	equivalence relation 2 classes	b equivalence relation ∞ classes

unidimensional

not multidimensional multidimensional

FIGURE 1

The "non-structure" results are proved in Chapters VII and VIII where Shelah establishes that if T is not superstable then T has 2^κ models of power κ for each uncountable κ. In spirit, the proof in the special case when T is unstable proceeds in the following way. First there are 2^κ linear orders I_α of power κ. Since T is unstable, T admits a formula which linearly orders some infinite sequence of n-tuples. Build by compactness a model containing a set of order indiscernibles (with respect to this formula) of type I_α. Let M_α be the Skolem hull of I_α. Now I_α may not be isomorphic to I_β while M_α is isomorphic to M_β. Start over, building linear orders sufficiently distinct that there are 2^κ distinct models among the M_α. The flesh requires a detailed combinatorial analysis to establish what sufficiently distinct linear orders are and that there are enough of them. The superstable case, which appears in this book for the first time, is similar in spirit, but even more complicated in the flesh. The linear orders must be replaced by trees, the notion of Skolem hull of a set of order indiscernibles must be generalized to the notion of a Skolem hull of a tree of indiscernibles, etc. These results differ from the structure results below as they hold for pseudo-elementary as well as elementary classes.

We will now discuss the positive "structure theory". Recall that a basis of a vector space can be described as either a maximal independent set or a minimal generating set. We want to describe a model by a sequence of cardinal numbers. These should be the cardinalities of maximal independent sets and, together, these sets should generate the model. Thus we need to generalize both the notions of independence and generation. The fact that we must consider more than one sort of independent set introduces complexities not even hinted at in vector spaces. We will first discuss one of these hidden difficulties, "What should we count?". Then we will consider the generalization of the notions of independence and generation. Finally, we will return to some of the further difficulties arising from consideration of families of independent sets. Some of these difficulties can be solved for the stable as well as the superstable case but for simplicity we will consider only the superstable case here.

If I is a set of algebraically independent elements in an algebraically closed field then every permutation, f, of I is an elementary map in the sense that for any formula $\phi(x_1, \ldots, x_n)$, $\models \phi(i_1, \ldots, i_n) \leftrightarrow \phi(f(i_1), \ldots, f(i_n))$. That is, in model theoretic parlance, I is a set of indiscernibles. It is natural, then, to try to characterize models by the cardinalities of maximal sets of indiscernibles. As soon, however, as we admit the possibility of needing two sets of indiscernibles (which is necessary in even so simple a case as the theory of an equivalence relation with two infinite classes) we must confront the problem of redundant information. For example, if we consider again the question of an equivalence relation, any two transversals yield the same information: the number of equivalence classes. Shelah solves this problem at the same time that he shows how to construct sets of indiscernibles and proposes a notion of dependence for elements of an arbitrary model.

Shelah defines the notion: $t(a; X)$ does not fork over A. This means intuitively that a depends no more on X than it does on A. In fact, if A is contained in B and C which are subfields of an algebraically closed field, then $t(C; B)$ does not fork over A means exactly that B and C are linearly disjoint over A. A type $p \in S(A)$ is stationary if it does not admit two contradictory extensions, neither of which forks over A. Now Shelah shows that if I is an independent set over A (i.e. for each $i \in I$ $t(i; A \cup I - \{i\})$ does not fork over A) and if all members of I realize the same stationary type then I is a set of indiscernibles. I is said to be based on that stationary type. Similarly, if $p \in S(A)$, $A_0 \subseteq A$, p does not fork over A_0 and $p|A_0$ is stationary then p is said to be based on A_0. We define $\kappa(T)$ as the least cardinal such that if $p \in S(A)$, then p does not fork over A_0 for some $A_0 \subseteq A$ with $|A_0| < \kappa(T)$. One sign of the tractability of superstable theories is that for superstable T, $\kappa(T) = \aleph_0$. Moreover, if T is superstable and I is an infinite set of indiscernibles then I is based on a finite subset I_0 of I. Finally, the type of redundant information mentioned above can be eliminated. For if I and J are based on the stationary types p and q then in any model M containing $I \cup J$, there are unique extensions p' and q' of p, q to members of $S(M)$ which do not fork over $\mathrm{dom}(p) \cup \mathrm{dom}(q)$. Calling $I \sim_1 J$ if $p' = q'$ resolves this difficulty with redundant information. However, it does so directly only at a price. That is, the procedure only works to find the dimension of indiscernible sets which are based in the model at hand. This is most easily resolved by restricting the problem to counting the $F^a_{\kappa(T)}$-saturatedn models (see below). Then all types over M are based in M. Since, even in the ω-stable case, this hypothesis requires that M be ω-saturated, it is not directly applicable for studying countable models.

The separation of the notions of independence and generation has the following further consequence. If we take a depends on X over B to mean $t(a; X \cup B)$ forks over B, then dependence is a notion of finite character and the familiar exchange principle holds. However, dependence is not transitive. Shelah deals with this by restricting himself to the class of elements of a model which realize what we calls regular types. He proves that a maximal independent set of such elements in a model M has a unique cardinality or dimension. However, (contrary to the claim of Theorem V.I.14) the full axioms of linear dependence i.e. transitivity, do not hold.

In algebra, one normally passes from a set to a model (e.g. a group) by closing the set under the algebraic operations. There is no such simple procedure for passing to an elementary submodel of M. Skolem functions can be added to regain a simple notion of generation but not without disturbing the spectrum of the theory. Morley replaced "M is generated by A" with "M is prime over A" (i.e. every elementary map from A into a model N of T can be extended to an elementary map of M into N). He showed that such prime model extensions exist for every A which is a substructure of an ω-stable theory. Shelah defines the notion of a model being prime within a class K of structures (e.g. κ-saturated models) and shows under what hypotheses one can prove the existence and uniqueness of K-prime models for various important classes K. One of these is what he calls the class of $F^a_{\kappa(T)}$-saturated models. We will call this notion strong saturation. M is strongly saturated if every type almost over (in a precise technical sense) a subset of M with fewer than $\kappa(T)$ elements is realized in M. In particular, this guarantees that every member of $S(M)$ is based in M.

Unfortunately, the notion of dependence described above does not always satisfy the transitivity axiom. Shelah defines a type to be regular if the notion above satisfies the properties of an abstract dependence relation on the realizations of p. He proves that if T is superstable each type has a well defined weight, a finite number of regular types on which it depends. This allows him to deduce the result (independently due to Lachlan [2]) that a countable superstable theory has either one or infinitely many countable models. In conjunction with the theorem described above, that a non-superstable theory has the maximum number of models in each uncountable power, this yields: If $1 < n(T, \aleph_0) < \aleph_0$ then $n(T, \kappa) = 2^\kappa$ for all uncountable κ. There remain many difficulties in generalizing this result to arbitrary stable T. Indeed there are several difficult open questions concerning countable models of stable but not superstable theories.

There is still a further difficulty with redundant information. Consider the theory of two equivalence relations E_1 and E_2 and suppose that each E_1 equivalence class intersects each E_2 equivalence class in exactly one element.

Then the model is determined by the number of E_2 equivalence classes and the cardinality of one of them. Nevertheless, two transversals for E_1 each contained in an E_2 class, yield inequivalent sets of indiscernibles in the sense that the canonical extensions of the types on which they are based are distinct. We described above an equivalence relation $\sim_1$ on sets of indiscernibles. Now there is a second equivalence relation $\sim_2$ defined by $I \sim_2 J$ if for every strongly saturated M every maximal $I' \subseteq M$ with $I' \sim_1 I$ and every maximal $J' \subseteq M$ with $J' \sim_1 J$, $|J'| = |I'|$. Note that $\sim_2$ can be viewed an equivalence relation on the $\sim_1$ equivalence classes.

Now, T is unidimensional if there is only one $\sim_2$ equivalence class. Naturally every $\aleph_1$-categorical theory is unidimensional and if T is ω-stable every unidimensional theory is $\aleph_1$-categorical. However, there are superstable unidimensional theories. The proof here of the generalized Łos conjecture (if T is $|T|^+$-categorical then T is categorical in all powers $> |T|$) proceeds by showing that such a theory is superstable and unidimensional and then by further analyzing categorical unidimensional theories.

Let T be a countable superstable theory and let $\aleph_\beta \geqslant c$, the power of the continuum. If the number of $\sim_2$ equivalence classes is unbounded then T is called multidimensional and T has at least 2^β models of cardinality $\aleph_\beta$. In contrast, if T is not multidimensional T has no more than $(\beta + 1)^c$ strongly saturated models of power $\aleph_\beta$. In fact, in this case a much more precise computation can be made because the following sharp theorem holds. For every strongly saturated model M, there is a strongly saturated model $N \subseteq M$ with $|N| \leqslant c$ and an independent set J such that M is prime among the strongly saturated models containing $N \cup J$. Note that there are theories which are neither multidimensional nor unidimensional.

We have summarised only about half the contents of the book. We have not mentioned the exhaustive treatment of rank in model theory nor the treatment of two-cardinal theorems; both subjects are essential to the lines we have discussed. In addition, there are major new results on ultraproducts and Keisler's order on theories which are peripheral to the main line.

The book contains no applications but we'll mention two which indicate the way this subject could influence algebra. First, the theory of separably closed fields of characteristic p is not superstable, indicating that a structure theory for such fields will be difficult despite the superficial resemblance to algebraically closed fields. Second, S. Garavaglia has shown that any ω-stable module can be uniquely decomposed into directly indecomposable modules. In addition to unifying a number of known results, this provides a new tractable class of modules for algebraists to study.

We have barely sketched the outline of Shelah's dimension theory. There are undoubtedly many refinements, simplifications and applications to come. Nevertheless, this book will remain the essential reference in the field for years to come.

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Singular homology theory, by William S. Massey, Graduate Texts in Math., Springer-Verlag, 1980, xii + 265 pp., $24.80.

Mathematics students normally encounter a mixture of courses on topics in pure mathematics, such as number theory or modern algebra, and courses in which mathematical techniques are applied to solve problems in the physical and biological sciences and engineering. But often the first occasion offering a student an opportunity to apply techniques from one branch of mathematics to solve theoretical problems in another is an introductory course in algebraic topology. Even for the student whose primary interest lies in another field, the subtle strength of these methods can be a source of genuine excitement and fascination. One need only spend some time trying to prove Brouwer's Theorem directly, that a continuous map of the closed n-disk D^n to itself must have a fixed point, to appreciate the effectiveness of homology theory.

These refined tools are not easily assimilated. The essential machinery must be constructed with care, for although the ideas may have solid geometric motivation, the level of abstraction and complexity can lead to confusion and a lost sense of direction. One must scrupulously avoid the tendency to view the constructions as the objective rather than as the tools to be applied. This transition is best made gradually, with ample explanations and examples. The patience and perseverance required are amply rewarded since a foundation is established for developing more sophisticated methods with applications far beyond these initial steps. Additionally, analogous constructions and techniques have evolved in other branches of mathematics and have become part of the established repertoire.

Perhaps these are in part the reasons why an introductory course in algebraic topology appears in the mathematical curriculum at many institutions. This has not always been the case. As recently as twenty-five years ago such courses were quite rare. The discipline itself was already well established, benefiting from the research of some of the finest mathematicians of the time. But most of these scholars were originally trained in related fields