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RESEARCH ANNOUNCEMENTS

ON THE INTEGRAL HOMOLOGY OF FINITELY-PRESENTED GROUPS

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1. In 1942 H. Hopf [5] pointed out that if G is a finitely-presented group then H_2G is finitely generated. (Here H_nG denotes the nth homology group of G with trivial integer coefficients.) Some 20 years later, in 1963, J. R. Stallings [10] constructed a finitely-presented group G with H_3G free abelian of infinite rank. Stallings' example suggested that the integral homology of finitely-presented groups was more complicated than had been suspected. The extraordinary complication of the subgroups of finitely-presented groups had already been revealed by a remarkable theorem of G. Higman [4] which joined recursive function theory to the theory of finitely-presented groups. Higman proved that a finitely-generated group is a subgroup of a finitely-presented group if and only if it can be defined by a recursively enumerable set of relations. In fact it follows from this theorem of Higman that finitely-presented groups of the type constructed by Stallings abound. The object of this announcement is to describe some theorems about the entire *integral homology sequence*

$$H_1G, H_2G, H_3G, \ldots$$

of a finitely-presented group G. In view of Higman's theorem it is not surprising that we couch these theorems in terms of recursive functions.

2. In order to do so we need to recall some terminology that is more or less in common usage. A presentation $(X; R)$ is termed *recursively enumerable,* or more briefly *r.e.* if

$$X = \{x_1, x_2, x_3, \ldots\}$$

is a countable set of generators and

$$R = \{r_1, r_2, r_3, \ldots\}$$

is an r.e. subset of the free group on X. We shall need also *abelian presentations* of abelian groups, which we denote by $(X; R)_{ab}$; here, of course, R is a subset of

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the free abelian group on X. The notion of an *r.e. abelian group presentation* is then defined analogously to that of an r.e. presentation. It follows without difficulty that an abelian group has an r.e. abelian group presentation if and only if it has an r.e. presentation. A sequence

$$(X_i; R_i)_{ab} \qquad (i = 1, 2, 3, \ldots)$$

of r.e. abelian group presentations is termed r.e. if $\bigcup_{i=1}^{\infty} R_i$ is an r.e. subset of the free abelian group on $\bigcup_{i=1}^{\infty} X_i$ (we have tacitly assumed that the X_i are disjoint). Similarly a sequence of abelian groups

$$A_1, A_2, A_3, \ldots$$

is termed an *r.e. sequence* if there are abelian group presentations

$$A_i = (X_i; R_i)_{ab} \qquad (i = 1, 2, 3, \ldots)$$

such that the resultant sequence of abelian group presentations is r.e.

3. The clue to understanding the integral homology sequence of a finitely-presented group is contained in the following theorem, which can be deduced with the aid of the so-called Gruenberg resolution [3].

THEOREM A. *The integral homology sequence of a finitely-presented group is an r.e. sequence in which the first two terms are finitely generated.*

Our major concern is with the converse to Theorem A. In order to explain our main result, we call an r.e. abelian group presentation $(X; R)_{ab}$ *untangled* if R is a basis of the subgroup it generates, and otherwise *tangled.* We are then able to prove

THEOREM B. *Let $A_1, A_2, A_3, \ldots$ be a sequence of abelian groups in which the first two terms are finitely generated. If each A_i has an untangled abelian group presentation and if this sequence of abelian group presentations is r.e. then there exists a finitely-presented group whose integral homology sequence is the given one.*

Now it turns out that there exists a finitely-presented group whose integral homology sequence is not an r.e. sequence of untangled presentations. So Theorem B is not the converse to Theorem A. However Theorem B, in conjunction with some untangling theorems, yields some results of interest in themselves.

4. The first of these untangling theorems is

THEOREM C. *Let $(X; R)_{ab}$ be an r.e. abelian group presentation of the torsion-free abelian group A. Then there is a recursive procedure whereby $(X; R)_{ab}$ can be transformed into an untangled presentation $(Y; S)_{ab}$ of A.*

Theorem B combines with Theorem C to yield

COROLLARY B1. *Let $A_1, A_2, A_3, \ldots$ be an r.e. sequence of torsion-free abelian groups. If A_1 and A_2 are finitely generated then there exists a finitely-presented group whose integral homology sequence is the given one.*

We have been unable to determine whether a torsion-free abelian group which has an r.e. presentation has an r.e. presentation with a solvable word problem. This then leaves unresolved the connection between Theorem C and our second untangling theorem:

THEOREM D. *Let $(X; R)_{ab}$ be an r.e. abelian group presentation with solvable word problem. Then there is a recursive procedure whereby $(X; R)_{ab}$ can be transformed into an untangled presentation $(Y; S)_{ab}$.*

Theorem D also combines with Theorem B to yield

COROLLARY B2. *Let $A_1, A_2, A_3, \ldots$ be an r.e. sequence of abelian groups with recursively solvable word problems. If A_1 and A_2 are finitely generated, then there exists a finitely-presented group whose integral homology sequence is the given one.*

A special case of Corollary B2 seems worth stating explicitly.

COROLLARY B3. *Let $A_1, A_2, A_3, \ldots$ be an r.e. sequence of finitely-generated abelian groups. Then there exists a finitely-presented group whose integral homology sequence is the given one.*

5. There are a number of ingredients that go into the proof of Theorem B. Perhaps the most important of these involves the notion of an acyclic group, i.e. a group G such that $H_n G = 0$ for every positive integer n. By combining Higman's theorem [4] with ideas in [1] we are able to prove

THEOREM E. *Every group with an r.e. presentation can be embedded in a finitely-presented acyclic group.*

Theorem E answers a question raised in [1].

Theorem B is proved by combining Theorem E with

THEOREM F. *Let A be an abelian group with an r.e. presentation. If $n \geqslant 3$, then there exists a finitely-presented group G such that*

$$H_n G \cong A.$$

In a sense Theorem F may be viewed as the ultimate generalisation of the example of Stallings cited at the outset.

In view of Theorem E it comes as no surprise that the homology of groups with an r.e. presentation can be approximated by finitely-presented groups.

THEOREM G. *Every group G with an r.e. presentation can be embedded in a finitely-presented group F in such a way that*

$$H_nG \cong H_nF \quad \text{for } n > 3.$$

In conclusion we would like to point out an amusing consequence of Theorem B which serves to underline its scope, viz.: there exists a finitely-presented group G such that H_nG is the cyclic group of order the nth integer in the decimal expansion of π.

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