

A SHORT PROOF OF THE DENJOY CONJECTURE

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1. In a recent letter Harold Shapiro has communicated to one of the authors his version of a nice (unpublished) proof of Matts Essén in which ideas from Heins' [2] version of the Denjoy-Carleman-Ahlfors theorem are applied to investigating the growth of certain univalent functions. It is the aim of the authors to show here that the direction may be reversed in a surprising way, giving a simple and natural proof of the Denjoy conjecture based on ideas from the theory of univalent functions. We wish to thank Harold Shapiro and Matts Essén for the inspiring communication.

Our proof of the D-C-A Theorem is based on a simple distortion theorem of a new type for univalent functions. This distortion theorem is interesting for its own right and has various extensions and applications, which we plan to publish at a later stage. Our distortion theorem may be proved either by path families methods or by inequalities for the logarithmic capacity. Although the former method gives sharper results and is also suitable for generalization to higher dimensions, we prefer the latter approach, because it is shorter and adequate for the proof of the

DENJOY-CARLEMAN-AHLFORS THEOREM. *If f is analytic in \mathbb{C} and has $n \geq 1$ distinct and finite asymptotic values, then $\liminf \log M(r)r^{-n/2} > 0$ as $r \rightarrow \infty$.*

2. **Stars.** By an n -star, $n \geq 1$, we mean a set S in \mathbb{C} , which is a union of n compact locally rectifiable Jordan arcs l_j , $1 \leq j \leq n$, called *arms*, with $l_j \cap l_k = \{0\}$ for $j \neq k$, such that $z = 0$ is the end point of all l_j . S is said to be *straight* if its arms are line segments, possibly of different lengths, but evenly spaced.

If g is a conformal map of $\hat{\mathbb{C}} \setminus S$ onto $\hat{\mathbb{C}} \setminus S'$ with $g(z) \rightarrow 0$ as $z \rightarrow 0$, we shall say shortly that g maps S conformally onto S' . If in addition $f(z) = z + \dots$ near $z = \infty$ we shall say that f is normalized.

3. **A distortion theorem.** (i) *Every n -star S , $n \geq 1$, can be mapped onto a straight n -star S' by a normalized conformal map.*

(ii) *Let S be an n -star, $n \geq 1$, and $r > 0$, such that $C(r) = \{z; |z| = r\}$*

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meets all arms of S . Let $D_j(r)$, $1 \leq j \leq n$, denote the n connected components of $D(r) \setminus S$, $D(r) = \{z: |z| < r\}$, with the property that $0 \in \overline{D_j(r)}$. If g is a normalized conformal map of S onto a straight n -star S' and $d_j = \text{Max}\{|g(z)|: z \in \overline{D_j(r)}\}$, then

$$\text{Min}\{d_j: 1 \leq j \leq n\} \leq (d_1 d_2 \cdots d_n)^{1/n} \leq cr, \tag{3.1}$$

where $c > 0$ is a constant which does not depend on r .

PROOF. Let Σ denote, as usual, the class of univalent functions

$$h(t) = t + b_0 + b_1/t + \cdots, \quad t \in \Delta = \{t: |t| > 1\}.$$

The proof of part (i) follows from the Riemann mapping theorem and the following known

LEMMA. (i) For any distinct n -points t_j , $1 \leq j \leq n$, on $\partial\Delta$ there is $h \in \Sigma$ which maps Δ onto the complement of a straight n -star such that $h(t) \rightarrow 0$ as $t \rightarrow t_j$, $1 \leq j \leq n$.

(ii) h is unique and given by

$$h(t) = t^{-1} \prod_{j=1}^n (t - t_j)^{2/n}. \tag{3.2}$$

The assertions of the Lemma may be verified by computing the change of $\arg h(t)$ as t varies along $\partial\Delta$, or by [1, p. 529].

The proof of part (ii) is based on some basic facts about the logarithmic capacity and on an important theorem of Pommerenke, a particular case of which was proved earlier by Schiffer. We first present the facts and then Pommerenke-Schiffer's theorem.

3.3. KNOWN FACTS ABOUT THE LOGARITHMIC CAPACITY. (i) If E is contained in a disc of radius r , then $\text{cap } E \leq r$.

(ii) If E is connected and $a, b \in E$, then $|a - b| \leq 4 \text{ cap } E$.

(iii) If E_j , $1 \leq j \leq n$, are compact and $E = \bigcup E_j$, then

$$\left[\prod_{k \neq j} \text{dist}(E_k, E_j) \right] \cdot \prod_j \text{cap } E_j \leq (\text{cap } E)^{n^2}.$$

For (i) and (ii) see [3, p. 336–337]. (iii) is an elementary known observation which appears as an exercise in [3, p. 341].

LEMMA (POMMERENKE [3, p. 346], SCHIFFER [4]). If $H \in \Sigma$, E is compact in Δ and $M = \text{Max}\{|t|: t \in E\}$; then

$$(\text{cap } E)^2 \leq M^2 \text{ cap } H(E). \tag{3.4}$$

PROOF OF THE DISTORTION THEOREM—CONCLUSION. Since g is continuous, $D_j(r)$ are connected and $g(z) \rightarrow 0$ as $z \rightarrow 0$, it follows that $D'_j(r) =$

$g(D_j(r))$ are connected and cluster at 0. Therefore, we can find compact connected sets E'_j in $D'_j(r)$ and points a'_j and b'_j in E'_j such that $|a'_j| = d_j/2$ and $|b'_j| = d_j/4$, $1 \leq j \leq n$. Let H be the Riemann mapping of Δ onto $\hat{C} \setminus S$. Since the normalization of g and (3.1) are not effected by replacing $g(z)$ by $\alpha g(z/\alpha)$ and S and S' by αS and $\alpha S'$, respectively, for $\alpha > 0$, we may assume that $H \in \Sigma$ and that $h \circ H$ is of the form (3.2). Let $E_j = h^{-1}(E'_j)$, $a_j = h^{-1}(a'_j)$, $b_j = h^{-1}(b'_j)$ and $E = \bigcup E_j$. Then by the nature of h and by virtue of 3.3(ii) it is easy to show that

$$c_1 d_j^{n/2} \leq |a_j - b_j|/4 \leq \text{cap } E_j, \quad 1 \leq j \leq n,$$

and therefore

$$\left(\prod d_j\right)^{1/n} \leq c_2 \left(\prod \text{cap } E_j\right)^{2/n^2} \leq c_3 (\text{cap } E)^2 \leq c_3 M^2 \text{cap } H(E) \leq cr,$$

where c and c_k , $k = 1, 2, 3$, are positive constants independent of r , which proves the assertion of (ii). Here, the second inequality follows from 3.3(iii), the third by (3.4) and the last one by 3.3(i).

4. Proof of the D-C-A Theorem. Let $F(z) = f(1/z)$ and $M_F(r) = \text{Max}|F(z)|$, $|z| = r$, $r > 0$. If the D-C-A Theorem is false, then $\log M_F(r_k) r_k^{n/2} \rightarrow 0$, as $k \rightarrow \infty$, for some sequence $r_k \rightarrow 0$. Since f has n distinct and finite asymptotic values, say a_j , $1 \leq j \leq n$, there is an n -star S with arms l_j such that $F(z) \rightarrow a_j$ as $z \rightarrow 0$ along l_j , $1 \leq j \leq n$. We may assume that every circle $C(r_k)$ meets all l_j .

Let $G = F \circ g^{-1}$, where g is a normalized conformal mapping of S onto a straight n -star S' . By the distortion theorem and by passing to a subsequence, denoted again by $\{r_k\}$, there is $j \in \{1, \dots, n\}$ such that $|g(t)| \leq c|z| = cr_k$, for all $z \in C(r_k) \cap \bar{D}_j(r_k)$ and $k = 1, 2, \dots$. Consider the infinite sector D'_j which contains $g(D_j(r_k)) = D'_j(r_k)$. Then $g(C(r_k) \cap \bar{D}_j(r_k))$ contains a connected subset $C'_j(r_k)$ which is contained in the closure of $D'_j \cap \{w: |w| \leq cr_k\}$, and which meets each of the two arms, say l'_j and l'_{j+1} of S' , which are contained in $\partial D'_j$.

The asymptotic values of f are finite and distinct, hence $G_j = G|_{D'_j}$ is bounded on l'_j and on l'_{j+1} and unbounded in D'_j , by Lindelöf's theorem on the asymptotic limits of bounded functions. Hence

$$M_{G_j}(cr_k) \leq \text{Max}\{|G(z)|, z \in C'_j(r_k)\} \leq M_F(r_k)$$

for all small r_k , by the maximum principle and by the relation between F and G . Thus

$$(cr_k)^{n/2} \log M_{G_j}(cr_k) \leq c^{n/2} r_k^{n/2} \log M_F(r_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

contradicting Phragmén-Lindelöf's theorem in D'_j . This completes the proof.

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