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# SYZYGIES OF SMALL RANK

BY E. GRAHAM EVANS AND PHILLIP GRIFFITH[1]

Let $R$ be a local domain and $M$ a finitely-generated $R$-module. The module $M$ is called a $k$th syzygy if it sits back $k$ steps in projective resolution of some $R$-module. The syzygy problem asks if nonfree $k$th syzygies of finite projective dimension necessarily have rank greater than or equal to $k$. We have established the following result.

THEOREM. *Let $R$ be a local Cohen-Macaulay domain containing a field and let $M$ be a finitely-generated $k$th syzygy of finite projective dimension and of rank less than $k$. Then $M$ is free.*

For $M$ an $R$-module and $m$ an element of $M$ we define the order ideal, $O_M(m)$, to be the ideal which consists of the images of $m$ under homomorphisms of $M$ to $R$. The proof proceeds by considering the height of particular order ideals, $O_M(m)$, in relation to the projective dimension of $M$.

Specifically, if $M$ is a $k$th syzygy of the $R$-module $N$ and if $x_1, \ldots, x_{k-1}$ is an $R$ sequence, then $\operatorname{Tor}_k^R(N, R/(x_1, \ldots, x_{k-1}))$ is zero. However, if $O_M(m)$ is contained in the ideal $(x_1, \ldots, x_{k-1})$, then considering the map from $M$ to the free module $F$ which occurs in the resolution of $N$, the image of $m$ is contained in $(x_1, \ldots, x_{k-1})F$. Thus $\operatorname{Tor}_k^R(N, R/(x_1, \ldots, x_{k-1}))$ is nonzero. However, if $O_M(m)$ is contained in the ideal $(x_1, \ldots, x_{k-1})$, then considering the map from $M$ to the free module $F$ which occurs in the resolution of $N$, the image of $m$ is contained in $(x_1, \ldots, x_{k-1})F$. Thus $\operatorname{Tor}_k^R(N, R/(x_1, \ldots, x_{k-1}))$ is nonzero. This idea is the key step in Gröbner's proof of Hilbert's syzygy theorem [4]. It shows that the ideal $O_M(m)$ tends to have height at least $k$ if $M$ is a $k$th syzygy. On the other hand the height of $O_M(m)$ tends to be bounded by the rank of $M$. While modules of small rank can have elements with the height of $O_M(m)$ large, we show that such phenomena cannot occur in the minimal counterexample to the syzygy problem.

The hypothesis that $R$ contains a field is needed to insure the existence of maximal Cohen-Macaulay modules [6] which replace the module $R/(x_1, \ldots, x_{k-1})$ in our modification of Gröbner's proof [4]. The hypothesis that $R$ is Cohen-Macaulay is needed to apply the Auslander-Bridger criterion [1] for a module to

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be a $k$th syzygy. In particular, $M$ is a $k$th syzygy of finite projective dimension over a Cohen-Macaulay ring $R$ precisely when $M$ is $S_k$. We recall that $M$ is $S_k$ means that $\operatorname{depth}_R M_P \geqslant \min(k, \text{height } P)$ for all prime ideals $P$ of $R$.

This result has several consequences in commutative algebra and algebraic geometry (cf. [2] for a more complete list). We mention two which seem among the most interesting.

COROLLARY 1. *Let $R$ be a regular local ring containing a field and let $P$ be a prime ideal of height two such that the module of dualizing differentials, $\Omega^0_{R/P}$, is cyclic. Then $R/P$ is a complete intersection. We note that $\Omega^0_{R/P}$ is cyclic if $R/P$ is a unique factorization domain.*

COROLLARY 2. *Let $E$ be an algebraic vector bundle on $\mathbf{P}^n$ of rank $k < n$ which is not a sum of line bundles. Then at least one of the twists of the cohomology groups $H^1(E), \ldots, H^{k-1}(E)$ is nonzero.*

This corollary is mentioned by Hartshorne [5]. Indeed it is equivalent to the original problem for the case of graded syzygies over rings of polynomials.

The preceding results will appear in our article [3].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, ILLINOIS 61801