DECIDABLE VARIETIES WITH MODULAR CONGRUENCE LATTICES

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ABSTRACT. For a large collection of varieties we show that if the first-order theory of such a variety is decidable then the variety decomposes into the product of two well-known highly specialized varieties. For many varieties the decidability question then reduces to a decidability question about modules.

A variety is a class of (abstract) algebras (belonging to some language) closed under the formation of direct products, subalgebras and homomorphic images. A variety \( V \) is locally finite if every finitely-generated member of \( V \) is finite. A class of algebras \( K \) generates a variety \( V \) if \( V \) is the smallest variety containing \( K \); then we say that \( V \) is generated by \( K \), written \( V = V(K) \). A variety is finitely generated if it is generated by finitely many finite algebras, or equivalently by a single finite algebra. The kernel of a homomorphism is called a congruence, and the congruences of any algebra form a lattice. A variety is congruence modular, or we prefer to say just modular, if the lattice of congruences of every algebra in the variety satisfies the modular law. (Most of the well-studied varieties are modular; for example varieties of groups, rings, modules and lattices are modular. However, the variety of semigroups is not modular.)

A variety \( V \) is a product of two subvarieties \( V_1, V_2 \) if \( V_1 \cup V_2 \) generates \( V \) and there is a term \( b(x, y) \) such that \( V_1 \models b(x, y) = x, \ V_2 \models b(x, y) = y \). If this is so we write \( V = V_1 \otimes V_2 \), and then for every algebra \( A \) in \( V \) there are (up to isomorphism) unique algebras \( A_1 \in V_1, A_2 \in V_2 \) such that \( A \cong A_1 \times A_2 \). A class \( K \) of first-order structures has a decidable theory if there is an effective procedure to determine precisely which first-order sentences are true of every member of \( K \).

A variety \( V \) is a discriminator variety if it is generated by a set \( K \) for which there exists a ternary term \( t(x, y, z) \) such that \( K \) satisfies \( t(x, y, z) = x \) if \( x \neq y; = z \) if \( x = y \). In everyday mathematics such varieties appear only as highly specialized varieties of rings, or varieties associated with algebraic logics.
The center $Z_A$ of an algebra $A$ is the binary relation defined by

$$(a, b) \in Z_A \iff \forall t \forall \tilde{u} \forall \tilde{v} \left[ t(\tilde{u}, a) = t(\tilde{v}, a) \iff t(\tilde{u}, b) = t(\tilde{v}, b) \right],$$

where $t$ denotes an $m + 1$-ary term and $\tilde{u}, \tilde{v}$ are $m$-ary sequences of variables, for any $m < \omega$. $Z_A$ is actually a congruence on $A$. An algebra is abelian if $V(A)$ is modular and $Z_A = A \times A$. Given any modular variety $V$ the abelian algebras in $V$ form a subvariety $V_{ab}$.

Let $BP^1$ denote the class of structures $(B, B_0, \lor, \land, ', 0, 1)$ where $(B, \lor, \land, ', 0, 1)$ is an atomic Boolean algebra and $(B_0, \lor, \land, ', 0, 1)$ is a subalgebra containing all the atoms of $B$.

**Theorem 1.** The theory of $BP^1$ is undecidable.

**Proof.** The class of finite graphs can be interpreted in $BP^1$ along the lines of a construction introduced by M. Rubin [6]. □

**Theorem 2.** If $V$ is a locally finite modular variety such that every reduct of $V$ to a finite language has a decidable theory, then $V$ has a subvariety $V_{ds}$ which is a discriminator variety and $V = V_{ds} \otimes V_{ab}$.

**Proof.** We focus our attention on three special kinds of finite algebras.

(Type I) $A$ is subdirectly irreducible and there is a subalgebra $B$ of $A$ and a congruence $\theta$ of $B$ such that $Z_A \cap (B \times B) < \theta < B \times B$.

(Type II) $A$ is subdirectly irreducible nonabelian with $Z_A = 0$, and there is an abelian subalgebra $B$ of $A$ with $|B| > 1$.

(Type III) $A$ is directly indecomposable but not simple and $V(A)$ is congruence distributive (i.e. every algebra in $V(A)$ has a lattice of congruences which satisfies the distributive law).

If $V$ contains an algebra $A$ of Type I or III then, using a modification of the Boolean power construction, we can interpret $BP^1$ into the class of subdirect powers of $A$. If $V$ contains an algebra $A$ of Type II then, extending a technique pioneered by Zamjatin [8] for the study of groups, finite bipartite graphs can be interpreted into the class of subdirect powers of $A$.

It follows that no finite member of $V$ can be of Type I, II or III. Then using A. Pixley’s characterization [5] of finitely-generated discriminator varieties plus the modular commutator introduced by J. Hagemann, C. Herrmann and J. D. H. Smith [3], [4], [7] and two results of R. Freese and R. McKenzie [2], [2a], one can prove that $V$ has the desired decomposition. □

**Theorem 3.** The question of ‘which modular varieties $V(A)$, generated by a finite algebra $A$ belonging to a finite language, have a decidable theory’ effectively reduces to the question of ‘which finite rings $R$ are such that the class of unitary left $R$-modules has a decidable theory’.
PROOF. Given a finite algebra $A$ of finite type one can effectively determine if $V(A)$ is modular, and also if $V(A)$ is of the form $V_{dS} \otimes V_{ab}$ where $V_{dS}$ is a discriminator variety and $V_{ab}$ is abelian. H. Werner (see [1]), extending results of S. Comer, proved that every finitely-generated discriminator variety belonging to a finite language has a decidable theory. If $V(A) = V_{dS} \otimes V_{ab}$ then one can effectively construct a finite ring $R$, given $A$, such that a decision procedure for the theory of $V_{ab}$ yields a decision procedure for the theory of unitary left $R$-modules, and vice-versa. □

REFERENCES

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