taken up in the exercises and historical notes. The notes are up to date. M. L. Wage's recent example of a normal space Z with $\operatorname{Ind} Z = 0$, $Z \times Z$ normal, and $\operatorname{Ind}(Z \times Z) > 0$ is mentioned as well as J. Walsh's infinite-dimensional compact metric space with no finite-dimensional subsets. (This is an improvement of D. Henderson's example which had no *closed* finite-dimensional subsets.)

From this book the student gets a good idea where dimension theory stands today. The lack of research questions indicates that this area may have passed its most fruitful period of research. The few remaining questions don't hold much prospect of giving us significantly new insights. New theorems and interesting examples will continue to appear, but it is unlikely that anything will arise to alter our basic perceptions of this theory. As with most theories which reach this state of maturity new ideas simply cannot find a place in the old theory. They must begin their life as a new theory and require a new classification.

Dimension theory is an excellent text giving us traditional dimension theory as it stands today. It presents all the essential features of interest to the general topologist without being compulsive. We have here a text that will probably be up to date for a considerable time.

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Representations of finite Chevalley groups, by Bhama Srinivasan, Lecture Notes in Math., vol. 764, Springer-Verlag, Berlin, Heidelberg, 1979, ix + 177 pp., $11.80.

The finite Chevalley groups are, roughly, the groups that arise when the real or complex parameters in a simple Lie group or, more generally, in a reductive one, are suitably replaced by the elements of a finite field. They include most of the finite simple groups, all except the alternating groups and the 26 "sporadic" groups, according to the classification which has just been completed. They thus occupy a central position in finite group theory. One of the important problems concerning them is the determination of their complex irreducible representations and characters. The first contribution here was made in 1896 by Frobenius **[1]** who determined the characters of the group $G = SL_2(k)$ over a finite field k. He first found the conjugacy classes of G, which is quite easy, and then built up the character table (a square matrix with rows indexed by conjugacy classes and columns by irreducible characters) by calculations not using much more than the orthogonality relations that this table was known to have. In 1907 Schur **[2]** redid Frobenius' work in a more conceptual way, obtaining many of the characters via concrete representations induced from one-dimensional representations of B, the group of upper-triangular matrices in G; but for those irreducible representations they cannot be obtained in this way, he, like Frobenius, could determine only the characters. This deficiency was soon noticed by others, since, e.g., the missing representations enter significantly into the work of Hecke [3] on automorphic forms, but was not overcome until many years later.

The situation then lay dormant until the 1940's when the characters of a handful of groups SL_3, GL_3, $SU_3, \ldots$ were found, mainly by the methods of Frobenius and Schur. Then in 1955 two major papers appeared. In the first, J. A. Green [4] determined the characters of the finite group $G = GL_n(k)$, introducing several ideas of great importance. First he showed that one could construct characters of G over $\mathbf{C}$ in terms of those over $\bar{k}$, such as those coming from the defining n-dimensional representation of G and its various exterior and symmetric powers and the like, by simply lifting the characteristic values of all matrices involved to suitable roots of 1 in $\mathbf{C}$. Not all complex characters can be obtained in this way, however, since, e.g., those obtained do not distinguish between unipotent classes whereas the set of all characters is known to do so. To compensate for this, Green inductively defined a set of polynomials in $|k|$ (now called Green polynomials), which, roughly, could be used to express the values of characters at arbitrary elements in terms of those at the semisimple (i.e. diagonalizable) ones. In the second major paper of 1955 Chevalley [5] introduced what are now called the Chevalley groups and developed their most important properties centering around the double coset decomposition relative to a Borel subgroup (described in more detail below). Soon after this, variations of Chevalley's construction were found yielding "twisted" Chevalley groups. The complete list of groups mentioned at the beginning was now in place as was a common framework for their further study, new even for the classical groups $(GL_n, SL_n, SO_n, SU_n, \ldots)$, which had, of course, already been studied for a long time.

To describe the situation in more detail we use the theory of algebraic groups which was developed at about the same time and has been an essential tool in our subject ever since. Let G be a connected reductive algebraic group over $\bar{k}$, the algebraic closure of a finite field k of q elements and of characteristic p. (We retain this notation throughout the review.) Thus G is a subgroup for some $GL_n(\bar{k})$ defined by polynomial relations on the matrix entries (e.g., SL_n: $\det = 1$), connected in the Zariski topology, and with its maximal connected unipotent normal subgroup (unipotent radical) trivial. Let F be an endomorphism of G such that G^F, the group of fixed points, is finite. The groups G^F that arise this way are, roughly, the Chevalley groups. The simplest example is $G = GL_n(\bar{k})$ with F the Frobenius morphism raising each matrix entry to its qth power. Then $G^F = GL_n(k)$. This group is not simple, but it becomes so, with a few exceptions, if we form the derived group and divide by the center. For $SU_n(k)$ one starts with $G = SL_n(\bar{k})$ and takes F to be the Frobenius composed with the inverse transpose map. Returning to our original group G, let B be a Borel subgroup (maximal connected solvable subgroup), T a maximal torus therein (isomorphic to GL_1^r for some r), N the normalizer of T, and $W = N/T$ the corresponding "Weyl group", a finite group about which a good deal is known. The essence of the Chevalley double coset decomposition referred to above is that a system of representatives for W is also one for $B \setminus G/B$; and the same holds for G^F in terms of

B^F and W^F if B and T are chosen to be invariant under F, as is always possible. For $G = GL_n$ or SL_n the standard choices are: B is the group of upper-triangular matrices, T the diagonal subgroup; then N consists of the monomial matrices (one nonzero entry in each row) and N/T is naturally isomorphic to S_n, the symmetric group of degree n. Also of importance in what follows is the notion of parabolic subgroup, one containing B (or a conjugate of such), and similarly for B^F. In GL_n, for example, there are 2^{n-1} such, one for each ordered partition $n = n_1 + n_2 + \cdots$, consisting of those elements of G that are upper-triangular in the corresponding block form. The resulting group P is not reductive, but it has the semidirect product decomposition $P = LV$ in which $L \sim GL_{n_1} \times GL_{n_2} \times \cdots$, consisting of the diagonal blocks, is reductive and V, the kernel of projection on L, is the unipotent radical. This suggests an inductive procedure, used by Schur and by Green in their work, for determining the representations of G^F: for each proper parabolic subgroup P fixed by F choose L to be fixed by F, pull the representations of L^F (known since L^F is smaller than G^F) up to P^F, then induce to G^F and decompose the results into their irreducible components; finally, find those representations ("discrete series") not coming from any P in this way. Of course, in the first step only the discrete series of L^F need be used since the other representations of L^F would come from its own proper parabolic subgroups. This process was made formal by Harish-Chandra in the late 1960's and applies also to reductive Lie groups and their representations.

But by then a good deal of work had already been done on the first problem, especially when $P = B$, and more especially when also the starting representation of L^F, hence also of B^F, is the trivial one, which we would like at least to mention in passing. Here the problem of decomposing the induced representation, R, into its irreducible components is, in principle, the same as that for $A(q)$, the commuting algebra of R. This "Hecke algebra" has, in terms of the basis consisting of the characteristic functions of the (B^F, B^F) double cosets of G^F, a multiplication table which is a deformed version of that of $\mathbf{Q}[W]$, depending on a parameter q (the number of elements of k in case F is the Frobenius). The study of the algebras $A(q)$ and their representations has been very fruitful not only in connection with the decomposition problem above and with representations of Weyl groups, but also in connection with representations of G^F in characteristic p and with infinite-dimensional representation of corresponding Lie groups (see **[6]** for some of the later developments).

We come now to the most important contribution to our subject so far, the great 1976 paper of P. Deligne and G. Lusztig **[7]**. Their idea is to realize the representations of G^F in suitable cohomology groups (or spaces) as has been done for Lie groups for many years. They use the l-adic cohomology with compact support. This attaches to each algebraic variety V in characteristic $p \neq 0$ groups $H^i(V)$ with coefficients in $\overline{Q}_l$ (l a prime different from p) or in some twisted version of this. This cohomology has many of the usual functorial properties, and these have entered into the solution of the Weil conjectures. The principal construction of Deligne-Lusztig is as follows. Let T

be a maximal torus of G fixed by F. (Such exist and are finite in number up to G^F-conjugacy; they correspond in case $W^F = W$, e.g., in case $G^F = GL_n(k)$ or $SL_n(k)$, to the conjugacy classes of W.) Let $B = TU$ ($U =$ unipotent radical) be a Borel subgroup containing T, and $L^{-1}(U)$ the set of all $x \in G$ such that $x^{-1}(Fx) \in U$, a closed subset of G covering U since by a theorem of Lang as extended by the reviewer every element of G has the form $x^{-1}(Fx)$. Even though $L^{-1}(U)$ need not be F-invariant (since B and U need not) the group $G^F \times T^F$ acts on it via $(g, t)x = gxt^{-1}$ and hence also on its alternating cohomology $\Sigma(-1)^i H^i(L^{-1}(U))$. The components of the latter, called $R_T^G(\theta)$, according to the action of T^F by its various characters θ, provide the virtual representations of G^F that are the principal objects of study of Deligne-Lusztig. The notation is justified since it turns out that, as G^F-module, $R_T^G(\theta)$ is independent of the choice of B above.

Before considering these representations in general we look at the construction in case $G = SL_2$ and F is the usual Frobenius. Here there are two classes of maximal tori. The first is represented by T, the subgroup of diagonal matrices. Here B consists of the matrices that are uppertriangular and U of those that are also unipotent, and both are fixed by F. By Lang's theorem applied to U, we have $L^{-1}(U) = G^F U$, a union of $|G^F/U^F|$ affine lines (copies of U), and G^F permutes these lines exactly as it does the elements of G^F/U^F. It follows that $R_T^G(\theta)$ is the θ-component of the representation of G^F induced by the trivial representation of U^F, i.e., is the representation of G^F induced by the representation θ of T^F pulled up to B^F. This case thus realizes Schur's first step, and for it no cohomological construction is really necessary. Consider, however, the second possibility $T_1 = gTg^{-1}$ (all maximal tori in G are conjugate) with g chosen so that $g^{-1}F(g)$, call it w, is in the normalizer of T, to make T_1 F-stable, but is not in T, to avoid the first case. We choose $U_1 = gUg^{-1}$ accordingly, and then $L^{-1}(U_1)g$ consists of all $x \in G$ such that $(xg^{-1})^{-1}F(xg^{-1}) \in gUg^{-1}$, i.e., $x^{-1}F(x) \in Uw$. If w, with its rows written out, is $[0, 1; -1, 0]$ and x is $[a, b; c, d]$, the last condition works out to $a = b^q$, $c = d^q$, whence $b^q d - bd^q = 1$ since $\det(x) = 1$. Thus the discrete series representations of $SL_2(k)$ which had eluded Frobenius, Schur and others for many years are here realized in the cohomology of this curve. Actually in the case of SL_2 the construction was first made by Drinfeld and this provided the insight into the general case.

We turn now to the general case. The principal tool used by Deligne-Lusztig to study the $R_T^G(\theta)$'s is the following result. Let X be an algebraic variety or scheme over $\bar{k}$ (as above), g an automorphism of X, and $\mathcal{L}(g, X) = \Sigma(-1)^i \operatorname{Tr}(g, H^i(X))$ the corresponding Lefschetz number. Then (a) $\mathcal{L}(g, X)$ is an integer independent of l. (b) If g is of finite order and s and u are its semisimple and unipotent parts (a p'-element and a p-element, respectively), then $\mathcal{L}(g, X) = \mathcal{L}(u, X^s)$. This, as well as the other nice properties that Lefschetz numbers tend to have, explains the use of alternating sums rather than individual cohomology groups in the original construction. The following results then flow. (c) The definition of the Green functions: $Q_T^G(u) = \operatorname{Tr}(u, R_T^G(1))$ (u unipotent). (d) A character formula expressing $\operatorname{Tr}(g, R_T^G(\theta))$ as a linear combination of values of θ at conjugates of s with coefficients that are values of Green functions of the connected centralizer of s. (e) Orthogonality relations: $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ are disjoint unless (T, θ) and (T', θ') are G-conjugate; their scalar product is 0 unless the pairs are G^F-conjugate in which case it is equal to the number of elements of $W(T)^F$ fixing θ; thus $R_T^G(\theta)$ is irreducible if this number is 1. (f) Orthogonality relations for the Green functions. (g) A formula for dim $R_T^G(\theta)$. (h) Completeness: a certain, relatively simple, rational linear combination of the $R_T^G(\theta)$'s equals the regular representation of G^F. (i) Connection with the Harish-Chandra program: If $(*)$ T is contained in a proper parabolic subgroup $P = LV$ fixed by F, then $R_T^G(\theta)$ equals $R_T^L(\theta)$ lifted to P^F and then induced to G^F. The irreducible representation R of G^F is in the discrete series if and only if it is disjoint from all $R_T^G(\theta)$'s with T as in $(*)$. In subsequent papers, Lusztig and others have carried these ideas forward, but the principal problem, that of decomposing the $R_T^G(\theta)$'s into their irreducible components, remains open in general.

Another approach to these results, at the character level, was initiated by T. Springer and completed by D. Kazhdan **[8]**, also in the 1970's. Here the Green functions are defined first, in terms of trigonometric sums on the Lie algebra of G; then certain class functions are defined by the character formula mentioned in (d) above, shown to be characters on G^F, and finally identified with the characteers of the $R_T^G(\theta)$'s. This development also uses the l-adic cohomology and requires p to be sufficiently large.

We come finally to the book under review. The author's purpose in writing it is to give a survey of the main recent developments in the subject, those that we have been discussing above, and to present an introduction to the l-adic cohomology which is palatable to finite group theorists. In the reviewer's opinion, she has succeeded admirably. In the book there is background material on algebraic groups, maximal tori, action of Frobenius and the like, and there are chapters on all of the topics that we have mentioned above as well as on some additional ones. In the chapter on the l-adic cohomology the author does the sensible thing by presenting the principal results axiomatically, but with full definitions, explanations and examples to make them intuitively plausible. In view of the new ideas and great activity in this area, the book is a timely one and the author is to be commended for bringing together all of the major threads of the subject in a very readable account.

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