BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 5, Number 2, September 1981

RESEARCH ANNOUNCEMENTS

ARTIN'S CONJECTURE FOR REPRESENTATIONS OF OCTAHEDRAL TYPE

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Let L/F be a finite Galois extension of number fields. E. Artin conjectured that the L-series of a nontrivial irreducible complex representation of $\mathrm{Gal}(L/F)$ is entire, and proved this for monomial representations. The nonmonomial two-dimensional representations are those with image in $PGL(2, \mathbf{C})$ isomorphic to the group of rigid motions of the tetrahedron, octahedron or icosahedron. In [5] Langlands proved Artin's conjecture for all two-dimensional representations of tetrahedral type and certain octahedral representations when $F = \mathbf{Q}$. The purpose of this note is to prove the conjecture for all octahedral representations by using the methods of Langlands and an analytic result of Jacquet, Piatetskii-Shapiro and Shalika.

Let ρ be an irreducible two-dimensional complex representation of $\mathrm{Gal}(L/F)$. We say that a cuspidal automorphic representation π of $GL(2, \mathbf{A}_F)$ equals $\pi(\rho)$ if $\pi = \bigotimes \pi_v$ with $\pi_v = \pi(\rho_v)$ in the sense of [2, §12] for almost all places v of F. When $\pi = \pi(\rho)$ the L-series of π and ρ agree, and since cuspidal representations have entire L-series, Artin's conjecture follows. In [5, §3] Langlands used base change for $GL(2)$ to produce candidates for $\pi(\rho)$. When ρ is octahedral we will use the following result to show that one of Langlands' candidates is in fact $\pi(\rho)$.

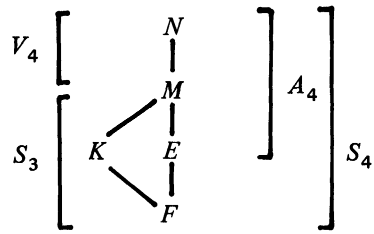
THEOREM [4]. *Let K be a cubic extension of F (not necessarily Galois). For each automorphic cuspidal representation π of $GL(2, \mathbf{A}_F)$ there exists an automorphic representation $\Pi = BC_{K/F}(\pi)$ of $GL(2, \mathbf{A}_K)$ such that for almost all places v of F, and each place w of K dividing v, $\pi_v = \pi(\sigma_v)$ implies that $\Pi_w = \pi(\mathrm{Res}^{W_{F_v}}_{W_{K_w}} \sigma_v)$.*

This theorem is proved using the theory of automorphic forms on $GL(3)$ and $GL(2) \times GL(3)$. The basic concept is similar to that of the example of quadratic base change given in [3, §20]. We recall that the theory of base change developed in [5] treats the case of Galois cyclic change of base of prime degree,

Received by the editors February 25, 1981.
1980 *Mathematics Subject Classification*. Primary 12A70; Secondary 10D40.

together with a characterization of the image and descent properties. We will denote by the symbol BC the base change lifting.

Let ρ be a two-dimensional representation of $\mathrm{Gal}(L/F)$ of octahedral type. Let E/F be the quadratic subextension of L/F fixed by all elements of $\mathrm{Gal}(L/F)$ mapping to the unique index two subgroup of the octahedral group S_4. Choose a 2-Sylow subgroup of the octahedral group and let K/F be the cubic subextension fixed by all elements of $\mathrm{Gal}(L/F)$ mapping to this chosen Sylow subgroup.



Let M be the compositium of E and K in L, so that we have the diagram of fields and Galois groups above. For any subextension T/F of L/F, let ρ_T be the restriction of ρ to $\mathrm{Gal}(L/T)$.

In [5, §3] Langlands showed, using base change and results of Gelbart, Jacquet, Piatetskii-Shapiro and Shalika, that $\pi(\rho_E)$ exists. There are exactly two cuspidal representations π_1 and π_2 of $GL(2, \mathbf{A}_F)$ such that $BC_{E/F}(\pi_i) = \pi(\rho_E)$. They are related by $\pi \approx \pi_2 \otimes \omega_{E/F}$, where $\omega_{E/F}$ is the idele class character corresponding to the quadratic extension E/F. Notice that ρ_K is monomial, so $\pi(\rho_K)$ exists.

LEMMA. *There exists a unique index i such that $BC_{K/F}(\pi_i) = \pi(\rho_K)$.*

PROOF. The theorem quoted above shows that $BC_{K/F}(\pi_i)$ exists for $i = 1, 2$. By transitivity of base change, $BC_{M/K}(BC_{K/F}(\pi_i)) = \pi(\rho_M)$ for $i = 1, 2$. Notice that $BC_{K/F}(\pi_2) \approx BC_{K/F}(\pi_1) \otimes \omega_{M/K}$. The representations $BC_{K/F}(\pi_i)$ are distinct for $i = 1, 2$, for if $BC_{K/F}(\pi_1) \approx BC_{K/F}(\pi_1) \otimes \omega_{M/K}$ then $\pi(\rho_M)$ would not be cuspidal. But ρ_M is irreducible, so this does not occur. By the descent theory for base change, $BC_{K/F}(\pi_1)$ and $BC_{K/F}(\pi_2)$ are the two automorphic representations of $GL(2, \mathbf{A}_K)$ which yield $\pi(\rho_M)$ after base change. Since $\pi(\rho_K)$ also has this property, it must be $BC_{K/F}(\pi_i)$ for a unique choice of i.

Let π be the automorphic representation π_i of the lemma which satisfies $BC_{K/F}(\pi) = \pi(\rho_K)$ and $BC_{E/F}(\pi) = \pi(\rho_E)$.

THEOREM. *Let ρ be an octahedral representation of* $\mathrm{Gal}(L/F)$. *Then $\pi(\rho)$ exists, and hence $L(\rho, s)$ is entire.*

PROOF. The proof is similar to [5, §3]. We show that the automorphic representation π constructed by Langlands is equal to $\pi(\rho)$. For each place v of

F such that ρ_v is unramified we obtain a diagonal conjugacy class $\operatorname{diag}(a_v, b_v)$ in $GL(2, \mathbf{C})$. For each place v of F such that π_v is an unramified principal series we obtain a conjugacy class $\operatorname{diag}(a'_v, b'_v)$. Let v be such that both ρ_v and π_v are unramified. Since $BC_{E/F}(\pi) = \pi(\rho_E)$ we see that $\operatorname{diag}(a'_v, b'_v)$ is conjugate to $\operatorname{diag}(a_v\omega, b_v\omega)$ with $\omega^2 = 1$. We must show that $\operatorname{diag}(a_v, b_v)$ and $\operatorname{diag}(a_v\omega, b_v\omega)$ are conjugate.

Let w be a place of K dividing v, with $[K_w: F_v] = d(w)$. Since $BC_{K/F}(\pi) = \pi(\rho_K)$, $\operatorname{diag}(a_v^{d(w)}, b_v^{d(w)})$ and $\operatorname{diag}((a_v\omega)^{d(w)}, (b_v\omega)^{d(w)})$ are conjugate. If $d(w) = 1$, the desired conjugacy results. If $d(w) = 3$, either $a_v^3 = a_v^3\omega$ or $a_v^3 = b_v^3\omega$. In the first case $\omega = 1$, while in the second $a_v = b_v\eta\omega$ with $\eta^3 = 1$. If $\eta = 1$, $\operatorname{diag}(a_v, b_v)$ is conjugate to $\operatorname{diag}(a_v\omega, b_v\omega)$. When η is nontrivial, $\operatorname{diag}(a_v, b_v) = \operatorname{diag}(b_v\eta\omega, b_v)$ gives an element of order 6 in the projective image of ρ. But the octahedral group contains no elements of order 6, so this is impossible.

Therefore, in all cases $\operatorname{diag}(a_v, b_v)$ is conjugate to $\operatorname{diag}(a_v\omega, b_v\omega)$. Since this holds for almost all places of F, we have $\pi = \pi(\rho)$, proving the theorem.

The icosahedral representations are not susceptible to these base change methods. Examples of icosahedral representations for $F = \mathbf{Q}$ which have entire L-series are given in [1].

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