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TWO WEIGHTS WITH ORTHOGONAL SUPPORTS BUT EQUAL ON A DENSE *-SUBALGEBRA

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ABSTRACT. We construct two normal semifinite weights φ and ψ on $\mathcal{B}(H)$ with orthogonal supports and such that $\varphi(x) = \psi(x)$ for x in a weakly dense *-subalgebra M_0 contained in $\mathfrak{M}_\varphi$ and $\mathfrak{M}_\psi$. The example is based on the existence of a pair of positive self-adjoint operators h and k on H with orthogonal supports and such that $\|h\xi\| = \|k\xi\|$ for ξ in a dense subspace $\mathcal{D}_0$ contained in $\mathcal{D}(h) \cap \mathcal{D}(k)$.

By a slight modification we obtain two commuting faithful normal semifinite weights on $\mathcal{B}(H)$ that agree on a weakly dense *-subalgebra. This shows that the condition of invariance of this subalgebra for the modular automorphism group may not be omitted in the theorem of Pedersen and Takesaki on equality of weights [2, Proposition 5.9].

Let M be a von Neumann algebra. If φ is a normal semifinite weight on M denote $\mathfrak{N}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}$ and $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi$. It is well known in the theory of weights that $\mathfrak{N}_\varphi$ is a left ideal and that $\mathfrak{M}_\varphi$ is a *-subalgebra spanned by its positive part $\mathfrak{M}_\varphi^+$ which is equal to $\{x \in M^+ \mid \varphi(x) < \infty\}$. The weight has a unique extension, which we still denote by φ, to a linear functional on $\mathfrak{M}_\varphi$. Because φ is assumed to be semifinite, the subalgebra $\mathfrak{M}_\varphi$ is weakly dense.

We will only be concerned with weights on the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space H. In the next proposition we will use the notation $\xi \otimes \eta$ for the rank one operator on H defined by $(\xi \otimes \eta)\zeta = \langle \zeta, \eta\rangle\xi$ whenever $\xi, \eta, \zeta \in H$.

1. PROPOSITION. *There is a one-to-one correspondence between the set of positive selfadjoint operators h on H and the set of normal semifinite weights φ on $\mathcal{B}(H)$ given by $\varphi(\xi \otimes \xi) = \|h\xi\|^2$ if $\xi \in \mathcal{D}(h)$ and $\varphi(\xi \otimes \xi) = \infty$ if $\xi \notin \mathcal{D}(h)$.*

This result essentially follows from the work of Pedersen and Takesaki [2] but can also be proved directly using a technique as in Lemma 1.4 of [1]. Moreover we have that φ is faithful if and only if h is nonsingular, and in that case the modular automorphisms are given by $\sigma_t(x) = h^{2it}xh^{-2it}$ for all $x \in \mathcal{B}(H)$ and all $t \in \mathbf{R}$.

To prove our main result we need a pair of positive selfadjoint operators with certain properties. The existence of such a pair was shown already in [3]. The proof is short and simple and we also give it here for completeness.

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2. PROPOSITION. *There exists a pair of positive selfadjoint operators h and k with orthogonal supports and such that* $\|h\xi\| = \|k\xi\|$ *for* ξ *in a dense subspace* $\mathcal{D}_0$ *contained in* $\mathcal{D}(h) \cap \mathcal{D}(k)$.

PROOF. Let a and b be two selfadjoint operators on a Hilbert space H_0 such that $\mathcal{D}(a) \cap \mathcal{D}(b) = \{0\}$. If we replace a and b by $a^*a + 1$ and $b^*b + 1$ we may assume, if necessary, that a and b are positive with bounded inverses satisfying $0 \leqslant a^{-1} \leqslant 1$ and $0 \leqslant b^{-1} \leqslant 1$.

Consider $H = H_0 \oplus H_0$ and define the operators h and k on H by

$$\mathcal{D}(h) = \{(\xi, \eta) | \xi \in \mathcal{D}(a), \eta \in H_0\},$$

$$\mathcal{D}(k) = \{(\xi, \eta) | \xi \in H_0, \eta \in \mathcal{D}(b)\}$$

and $h(\xi, \eta) = (a\xi, 0)$ if $(\xi, \eta) \in \mathcal{D}(h)$ and $k(\xi, \eta) = (0, b\eta)$ if $(\xi, \eta) \in \mathcal{D}(k)$. Then h and k are positive selfadjoint operators with orthogonal supports.

If we also let $\mathcal{D}_0 = \{(a^{-1}\eta, b^{-1}\eta) | \eta \in H_0\}$ then $\mathcal{D}_0 \subseteq \mathcal{D}(h) \cap \mathcal{D}(k)$ and $\|h\xi\| = \|k\xi\|$ for all $\xi \in \mathcal{D}_0$. To prove that $\mathcal{D}_0$ is dense we assume that (η_1, η_2) is a pair in H such that $(\eta_1, \eta_2) \perp (a^{-1}\eta, b^{-1}\eta)$ for all $\eta \in H_0$. Then $\langle a^{-1}\eta_1 + b^{-1}\eta_2, \eta\rangle = \langle \eta_1, a^{-1}\eta\rangle + \langle \eta_2, b^{-1}\eta\rangle = 0$ for all $\eta \in H_0$ so that $a^{-1}\eta_1 = -b^{-1}\eta_2$. Because $\mathcal{D}(a) \cap \mathcal{D}(b) = \{0\}$ it follows that $a^{-1}\eta_1 = b^{-1}\eta_2 = 0$ and $n_1 = n_2 = 0$. This completes the proof.

It is now easy to obtain our main result.

3. THEOREM. *There exists a pair of normal semifinite weights* φ *and* ψ *on* $\mathcal{B}(H)$ *with orthogonal supports such that* $\varphi(x) = \psi(x)$ *for* x *in a weakly dense *-subalgebra* M_0 *of* $\mathcal{B}(H)$ *contained in* $\mathfrak{M}_\varphi$ *and* $\mathfrak{M}_\psi$.

PROOF. Choose a pair of positive selfadjoint operators h and k on H with orthogonal supports but such that $\|h\xi\| = \|k\xi\|$ for all ξ in a dense subspace $\mathcal{D}_0$ contained in $\mathcal{D}(h) \cap \mathcal{D}(k)$. Let φ and ψ be the corresponding weights on $\mathcal{B}(H)$.

If e is the support projection of h then $(1 - e)H \subseteq \mathcal{D}(h)$ and $h(1 - e) = 0$. And if $\{\xi_\alpha | \alpha \in A\}$ is an orthonormal basis for H we get

$$\varphi(1 - e) = \sum_{\alpha \in A} \varphi((1 - e)\xi_\alpha \otimes (1 - e)\xi_\alpha) = \sum_{\alpha \in A} \|h(1 - e)\xi_\alpha\|^2 = 0$$

so that the support of φ is contained in e. Therefore φ and ψ will have orthogonal supports. Now let M_0 be the algebra of linear combinations of the operators $\{\xi \otimes \eta | \xi, \eta \in \mathcal{D}_0\}$. If $\xi \in \mathcal{D}_0$ then $\varphi(\xi \otimes \xi) = \|h\xi\|^2 = \|k\xi\|^2 = \psi(\xi \otimes \xi)$ and by linearity and some kind of polarization we obtain $\varphi(x) = \psi(x)$ for all $x \in M_0$. And finally because $\mathcal{D}_0$ is dense in H also M_0 will be norm dense in the algebra of finite rank operators, and so weakly dense in $\mathcal{B}(H)$.

Pedersen and Takesaki have considered the problem of equality of two normal semifinite weights when they coincide on a dense *-subalgebra. They showed that φ and ψ are equal if they commute (i.e. if ψ is invariant for the modular automorphisms of φ) and if they agree on a dense *-subalgebra which is invariant under the modular automorphisms of φ. Our next result shows that the invariance of the subalgebra cannot be omitted, and only a small modification to the example in Theorem 3 has to be made.

4. THEOREM. *There exists a pair of different faithful normal semifinite weights φ and ψ on $\mathcal{B}(\mathcal{H})$ such that ψ is invariant for the modular automorphisms of φ and such that $\psi(x) = \varphi(x)$ for x in a weakly dense *-subspace $\mathcal{M}_0$ contained in $\mathfrak{M}_\varphi \cap \mathfrak{M}_\psi$.*

PROOF. Again let h and k be as in Proposition 2. Then consider $h_1 = (h^2 + 1)^{1/2}$ and $k_1 = (k^2 + 1)^{1/2}$ and let φ and ψ be the weights associated to h_1 and k_1 respectively. The weights will be faithful as now the operators are nonsingular. Because h and k have orthogonal supports it also follows easily that h_1 and k_1 will commute. Then ψ will be invariant for the modular automorphisms of φ given by $\sigma_t(x) = h_1^{2it} x h_1^{-2it}$ when $x \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbf{R}$. We also get $\|h_1\xi\|^2 = \|h\xi\|^2 + \|\xi\|^2 = \|k\xi\|^2 + \|\xi\|^2 = \|k_1\xi\|^2$ for any $\xi \in \mathcal{D}_0$ and so if as before $\mathcal{M}_0$ is the algebra spanned by the operators $\xi \otimes \eta$ with $\xi, \eta \in \mathcal{D}_0$ then $\psi(x) = \varphi(x)$ for all $x \in \mathcal{M}_0$.

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